CE 544
Advanced Material Science

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Chapter 5
Stress – Strain Relationships and Behavior

Objectives

- Become familiar with the elastic, plastic, steady creep and transient creep types of strain, as well as simple rheological models for representing the stress-strain-time behavior for each.
- Explore 3-D stress-strain relationships for linear-elastic deformation in isotropic materials, analyzing the interdependence of stresses or strains imposed in more than one – direction
- Extend the knowledge of elastic behavior to basic cases of anisotropy, including sheets of matrix-and-fiber composite material.

Chapter 5: Stress – Strain Equations and Models

✓ Three major types of deformation - Elastic, plastic, creep

1. Elastic deformation

   Associated with stretching but not breaking of chemical bonds

2. Plastic deformation

   Atoms change their relative positions, such as slip of crystal planes or sliding of chain molecules

3. Creep deformation

   Atoms change their relative positions, such as slip of crystal planes or sliding of chain molecules

Time independent (1–2)

Time dependent (3)

Stress – Strain Equations or Constitutive Equations

In engineering design and analysis, equations describing stress-strain behavior.

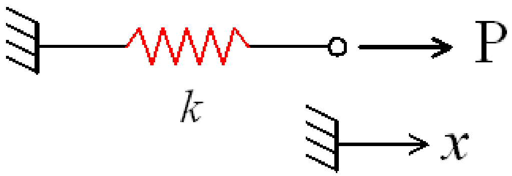
Rheological Models

Simple mechanical devices such as linear springs, frictional sliders and viscous dashpots can be used as an aid to understanding the various types of deformation.

⬇

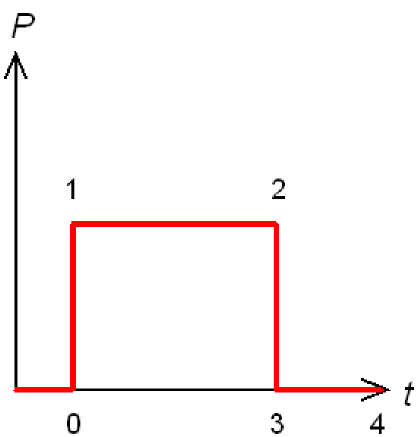
Such devices and combinations of them are called ***rheological models.***

5.2 Models for deformation behavior
Elastic Deformation

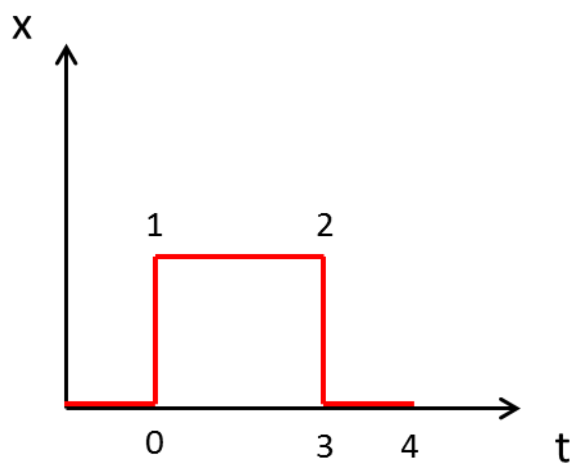


$P = kx \; (\sigma = E\varepsilon)$

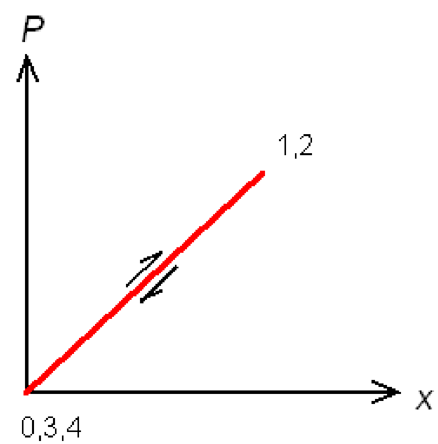
Input



x-t Response



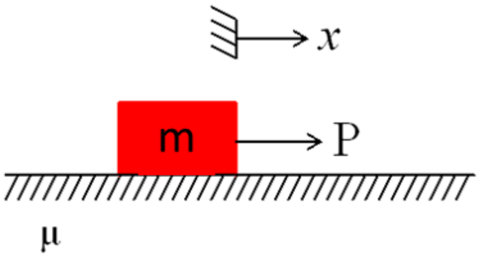
P-x path



0-1 Loading
1-2 Load is held constant
2-3 Unloading
3-4 No load

- Deformation is proportinal to the force (k = P/x)
- and recovered instantly upon unloading.

5.2 Models for deformation behavior
Plastic Deformation

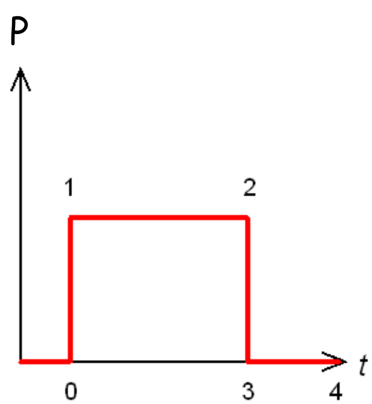


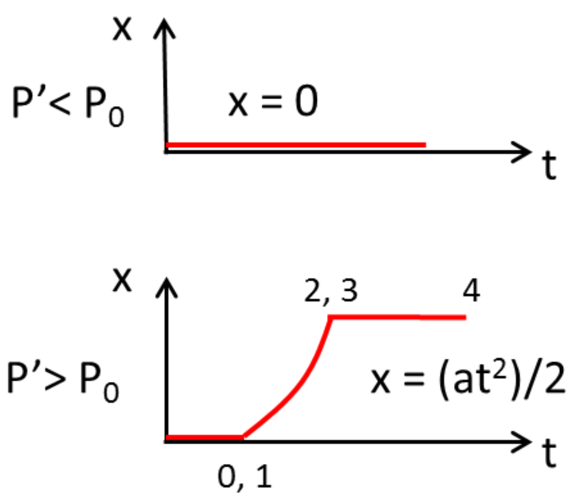
$x = 0$ if $|P| < P_o$
($\varepsilon = 0$ if $|\sigma| < \sigma_o$)

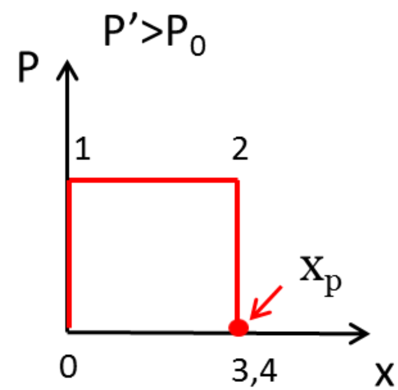
Input

x-t Response

P-x path

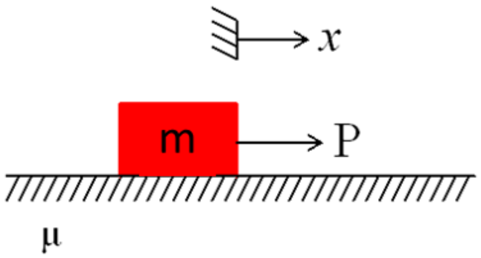






0-1 P_o must be exceeded
1-2 Load is held constant
2-3 Load is decreased to 0.

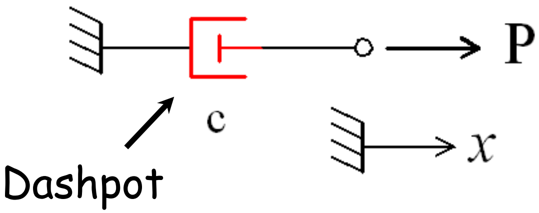
5.2 Models for deformation behavior
Plastic Deformation, cont'd



x = 0 if | P | < P_o
($\varepsilon = 0$ if $|\sigma| < \sigma_o$)

- Po is the critical force for motion
- If P' > P_0, the block moves with an acceleration $a = (P' - P_0)/m$
- When the force is removed at time t, the block has moved a distance $x=(at^2)/2$, and it remains at this new location.

5.2 Models for deformation behavior
Steady-State Creep

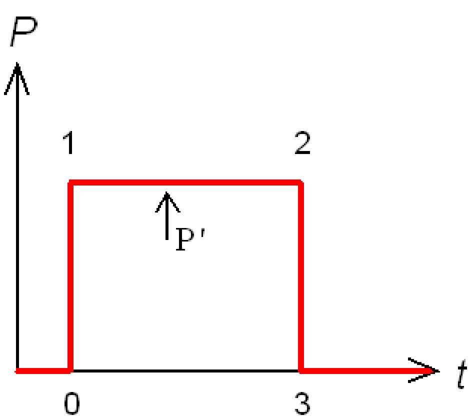


$$P = c\dot{x}$$

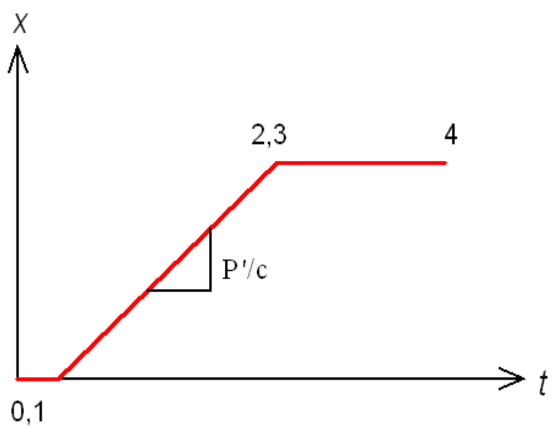
$$(\sigma = \eta\dot{\varepsilon})$$

Dashpot → a piston in a cylinder filled with a viscous fluid, such as a heavy oil.

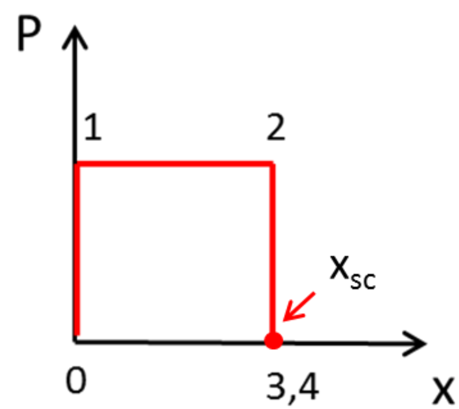
Input



x-t Response



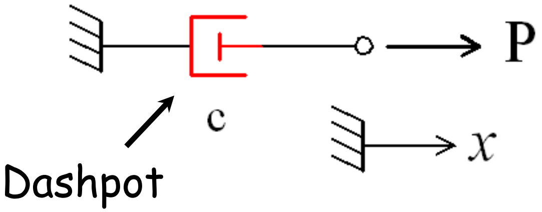
P-x path



0-1	loading
1-2	constant load
2-3	Instant unloading

5.2 Models for deformation behavior
Steady-State Creep, cont'd

Model



$$P = c\dot{x}$$

$$(\sigma = \eta\dot{\varepsilon})$$

- Proceeds at a constant rate of deformation under constant stress.
- Velocity ($\dot{x}$ = dx / dt) is proportional to the force.
- "c" is the dashpot constant so that a constant value of force P' gives a constant velocity. $\dot{x}$ = P' /c resulting in a linear displacement vs. time behavior.

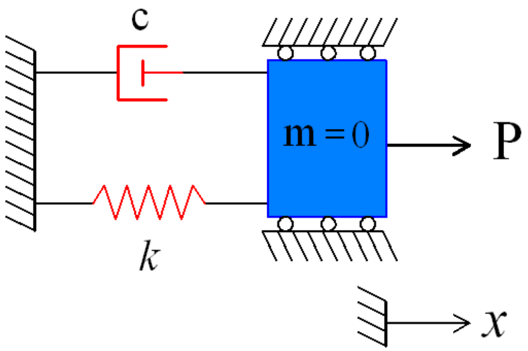
Steady-State Creep, cont'd

What happens when the force is removed?

- ✓ The motion stops, so that the deformation is permanent, that is not recovered.
- ✓ When a force is applied → small amounts of oil leak past the piston, allowing the piston to move.

Velocity of motion will be applied proportional to the magnitude of the force.

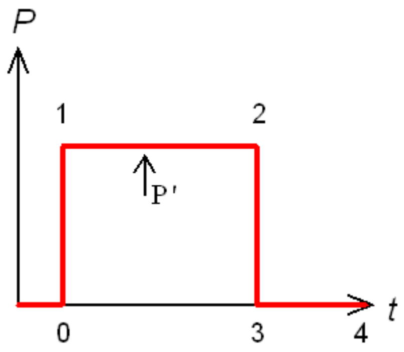
5.2 Models for deformation behavior
Transient Creep



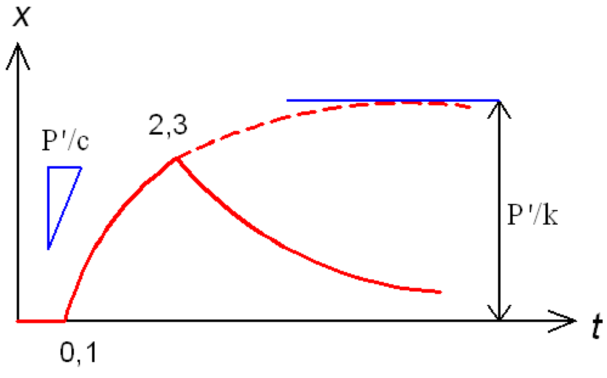
$$(P = kx + c\dot{x})$$

$$(\sigma = E\varepsilon + \eta\dot{\varepsilon})$$

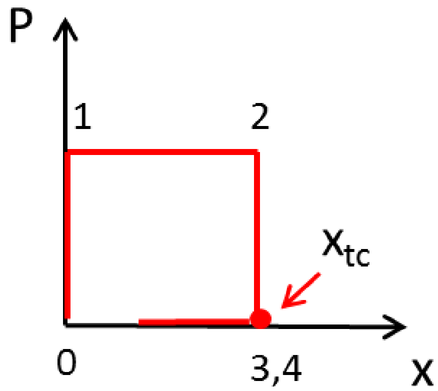
Input



x-t Response



P-x path



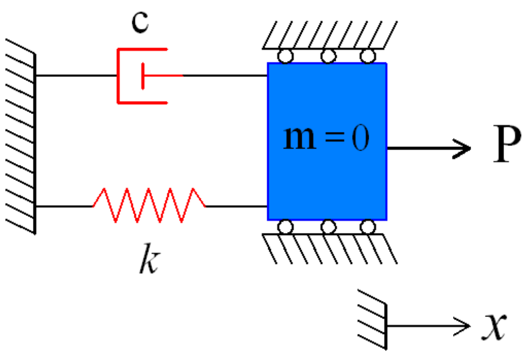
0-1 Loading (Inst.)

1-2 Constant Load

Deformation increases with time

2-3 Unloading

5.2 Models for deformation behavior
Transient Creep, cont'd



$$(P = kx + c\dot{x})$$

$$(\sigma = E\varepsilon + \eta\dot{\varepsilon})$$

This type of creep slows down as the time passes!

- An ***increasing*** fraction of the applied load is needed to pull against the spring as ***x increases***, so that ***less force*** is available to the dashpot and the rate of ***deformation decreases***.
- Deformation approaches to P'/ k if the force is maintained for a long period of time.

What happens when the force is removed?

✓ The spring, having been extended, pulls against the dashpot and all deformation recovers with time.

Comparison of deformations yielded by different models

Elastic → Time ***independent*** deformations ***recovered***.

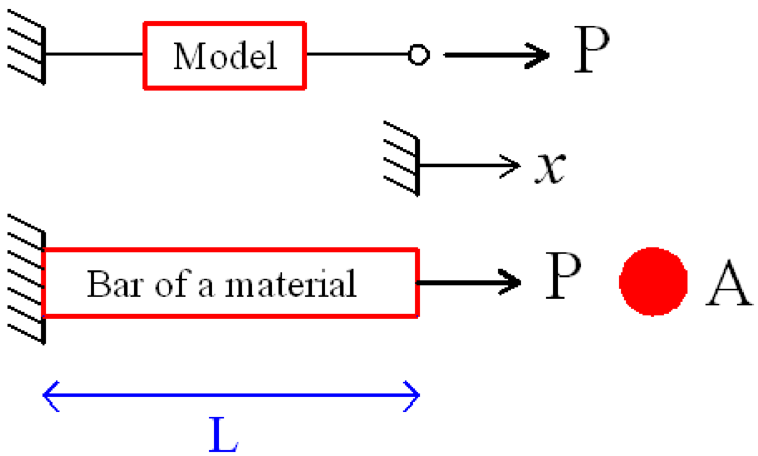
Plastic → Time ***independent*** deformations are **not recovered**.

SSC → Time ***dependent*** deformations are ***not recovered***.

TC → Time ***dependent*** deformations are **recovered**.

Example:

Relationship of models to stress, strain and strain rate in a bar of materials.



$$Stress = \sigma = P/A$$

$$Strain = \varepsilon = x/L$$

$$d\varepsilon/dt = \dot{\varepsilon} = \dot{x}/L$$

Model constants are related to material constants that are ***independent*** of the bar length L or area A.

$$\left.\begin{array}{l} \sigma = E\varepsilon \\ \varepsilon = x/L \end{array}\right\} \frac{P}{A} = \frac{Ex}{L} \Rightarrow \frac{kx}{A} = \frac{Ex}{L} \Rightarrow E = \frac{kL}{A} \quad \Big\rangle$$

For elastic deformations

Example, cont'd

The material constant analagous to dashpot constant "c" is called the coefficient of tensile viscosity.

$$\eta = \frac{\sigma}{\dot{\varepsilon}} \qquad \dot{\varepsilon} \rightarrow strain\ rate$$

$$P = c\dot{x}$$

$$\eta = \frac{\overbrace{c\dot{x}/A}^{\sigma}}{\dot{x}/L} \Rightarrow \eta = \frac{cL}{A}$$

For the plastic deformations, the yield strength of the model:

$$\sigma_o = \frac{P_o}{A}$$

5.2.1 Plastic Deformation Models

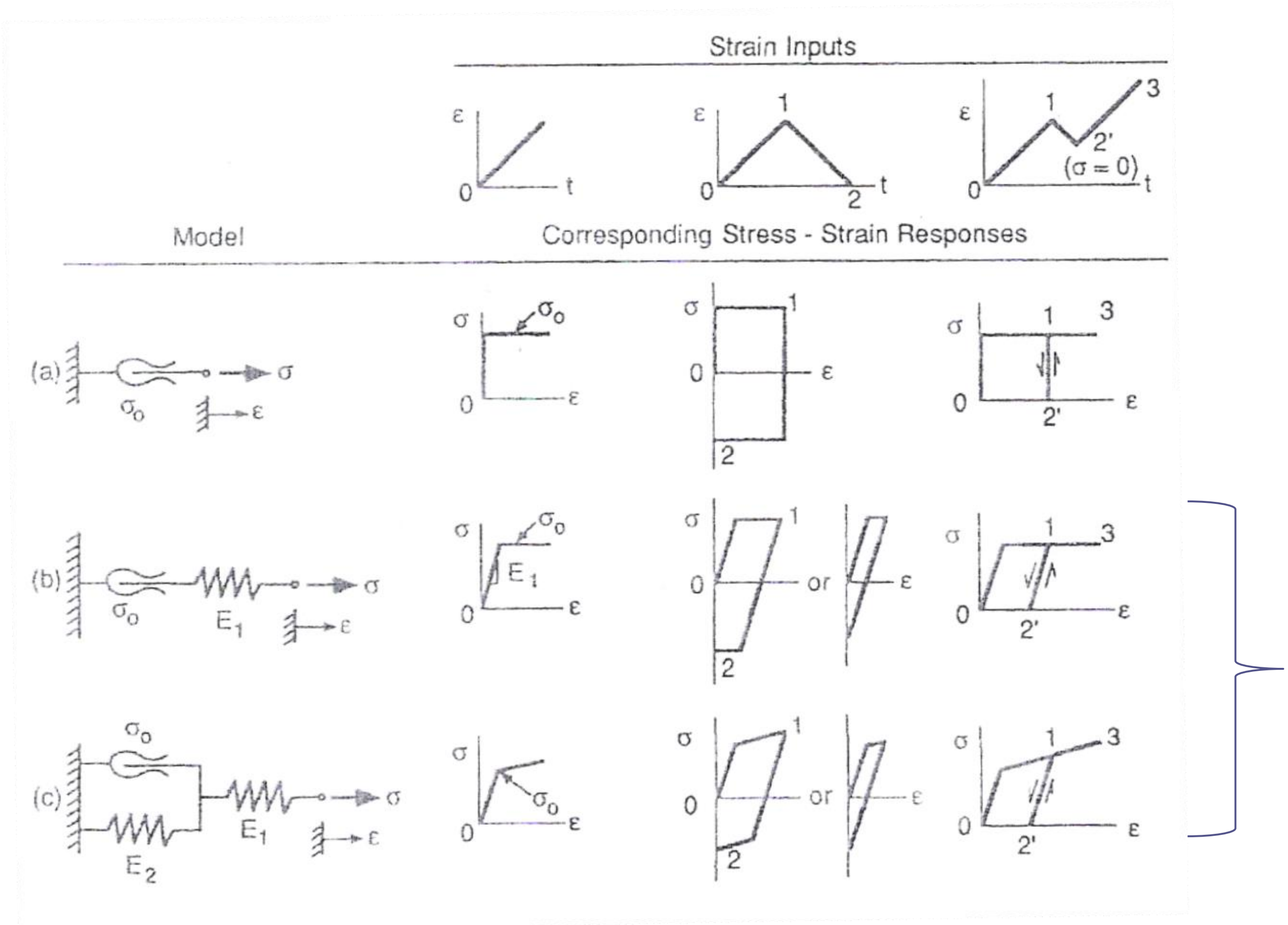
- ✓ **Principal mechanism causing plastic deformations in metals and ceramics;** sliding (slip) between planes of atoms in the crystal grains of the material occurring in an incremental manner due to dislocation motion.

Material`s resistance to plastic deformation ANALOGOUS TO Friction of a block on a plane!

Spring clip; has "0" mass but identical force vs. displacement response for slow loading.

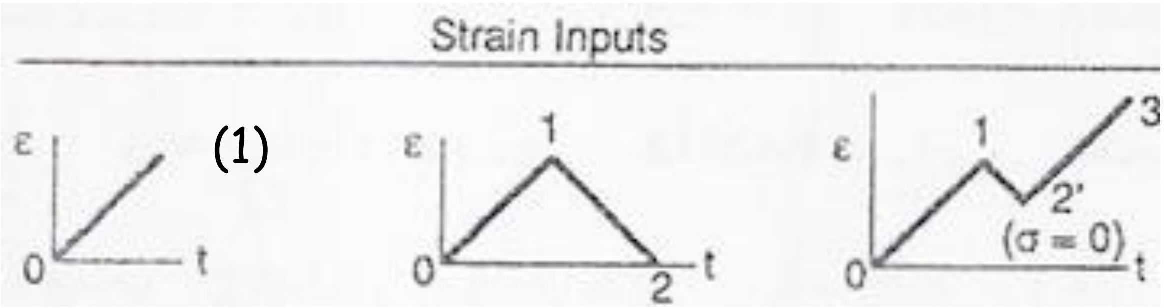
5.2.1 Plastic Deformation Models, cont'd

Rheological models for plastic deformation and their responses to three different strain inputs. Model (a) rigid, perfectly plastic, (b) elastic, perfectly plastic and (c) elastic, linear hardening.



Improved representation of the behavior of real materials.

5.2.1 Plastic Deformation Models, cont'd



(1) Simple monotonic straining → straining in a single direction

For this loading, for models (a) and (b) the stress remains at σ_o beyond yielding.

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad \varepsilon_1 = \sigma / E_1$$

For monotonic loading of model (c), the strain ε is the sum of strain ε_1 in spring E_1 and strain ε_2 in the (E_2, σ_o) parallel combination.

The vertical bar is assumed not to rotate, so that both spring E_2 and slider σ_o have the same strain. Prior to yielding, the slider prevents motion so that strain ε_2 is zero.

$$(\sigma \leq \sigma_o) \rightarrow \varepsilon_2 = 0, \quad \varepsilon = \sigma / E_1$$ (Before/prior to yielding)

Stress at E_2 = ?

Slider has a constant stress σ_o (so that the stress in spring E_2 is $(\sigma - \sigma_o)$). Hence the strain ε_2 and the overall strain ε are calculated as given below.

$$(\sigma \geq \sigma_o) \rightarrow \varepsilon_2 = \frac{\sigma - \sigma_o}{E_2} \qquad \varepsilon = \frac{\sigma}{E_1} + \frac{\sigma - \sigma_o}{E_2}$$

(Beyond Yielding)

The slope of stress-strain curve is seen to be;

$$\varepsilon = \frac{\sigma E_2 + (\sigma - \sigma_o) E_1}{E_1 E_2} \quad \Rightarrow \quad \varepsilon E_1 E_2 = \sigma E_2 + \sigma E_1 - \sigma_o E_1$$

$$\frac{d}{d\varepsilon}(\varepsilon E_1 E_2) = \frac{d}{d\varepsilon}(\sigma (E_2 + E_1)) - \frac{d}{d\varepsilon}\sigma_o E_1$$

$$E_1 E_2 = \frac{d\sigma}{d\varepsilon}(E_2 + E_1)$$

$$\frac{d\sigma}{d\varepsilon} = \underbrace{E_e}_{\substack{Equivalent \\ stiffness}} = \frac{E_1 E_2}{E_2 + E_1}$$

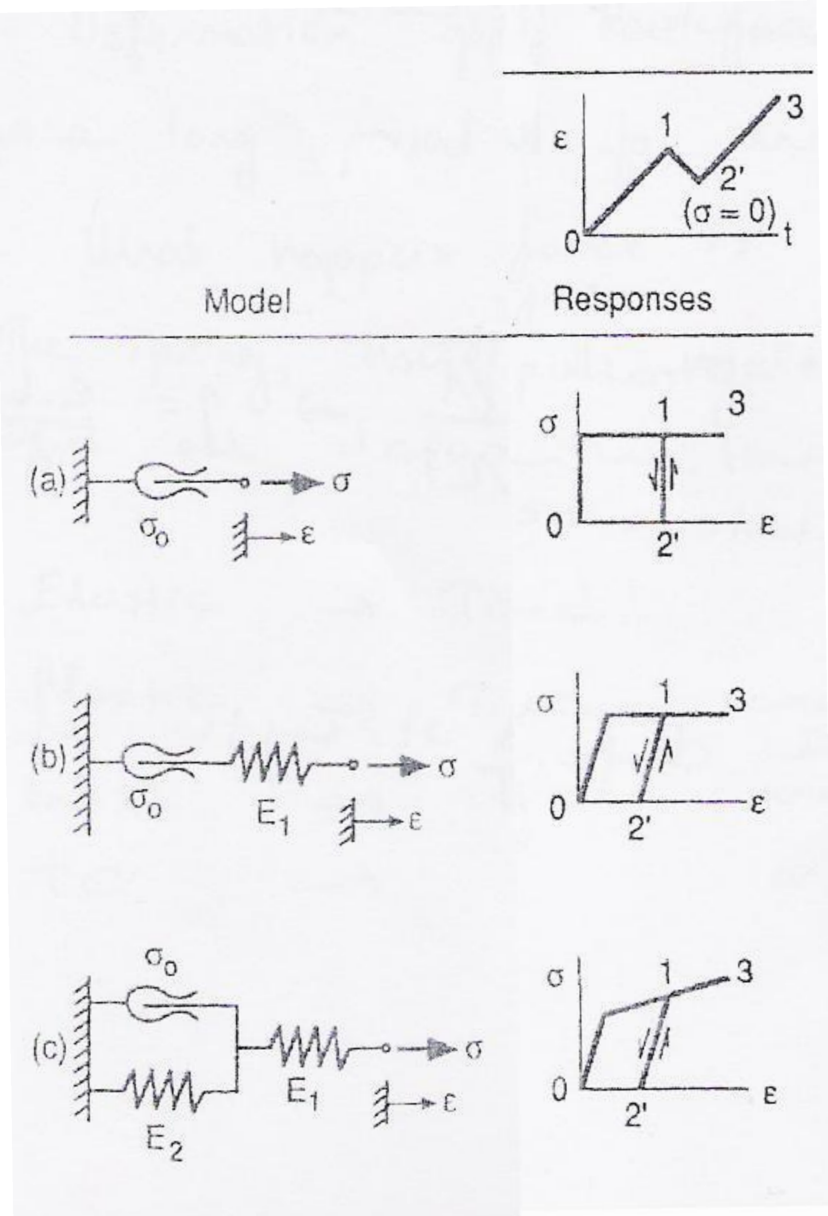
Equivalent stiffness corresponding to E_1 and E_2 in series is lower than both E_1 and E_2.

5.2.1 Plastic Deformation Models, cont'd

In 2nd and 3rd modes of straining,

- ✓ Strain is increased beyond yielding and then decreased to zero.
- ✓ In all 3 cases, there is no additional motion in the slider until the stress has changed by an amount $2\ \sigma_0$ in the negative direction.

(for b and c) ε_e → corresponds to the relaxation of spring E_1. Permanent or plastic strain ε_p corresponds to the motion of the slider up to the point of maximum strain.



Material is reloaded after elastic unloading to σ=0

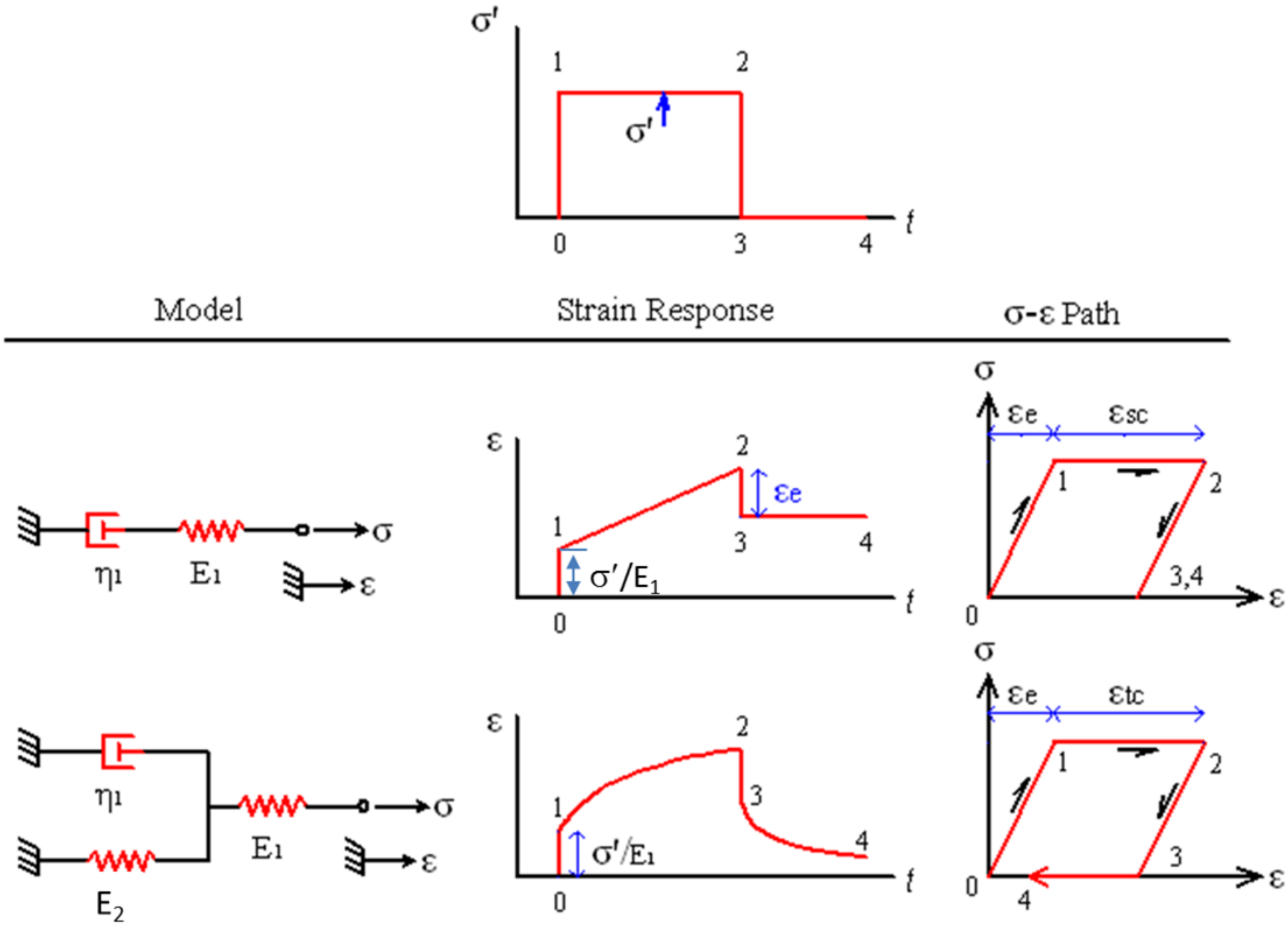
- In all cases yielding occurs a second time when the strain again reaches the value ε_1 from which unloading occurred.

→ 2 plastic models again yield at σ_o

→ Linear hardening model yields at $\sigma = \sigma_1$, which is higher than the initial yield stress. σ_1 is the same value of stress that was present at $\varepsilon = \varepsilon_1$, when the unloading first began. For all models it can be said that the model possesses a memory of the point of previous unloading.

<u>Real materials that deform plastically exhibit a similar memory effect.</u>

5.2.2. Creep deformation models



a) Steady-state creep with elastic spring added

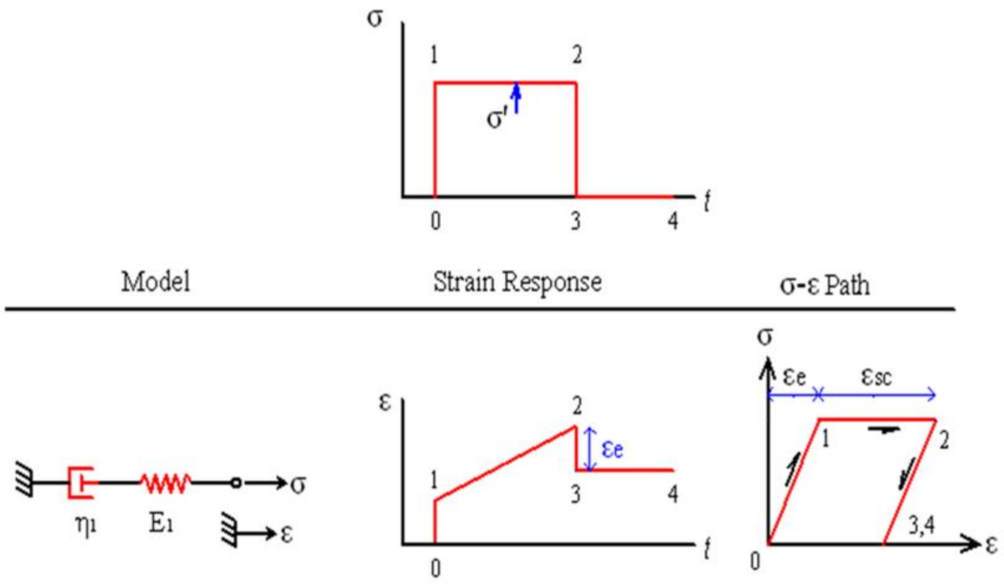
b) Transient creep with elastic spring added

5.2.2. Creep deformation models, cont'd

$$\text{Spring deform according to } \varepsilon = \sigma / E$$
$$\text{and dashpot according to } \dot{\varepsilon} = \sigma / \eta$$

- Constant viscosities are used in these models and this results in all strain rates and strains being proportional to the applied stress, a situation described by the term linear viscoelasticity. SUCH IDEALIZED BEHAVIOR is SOMETIMES A REASONABLE APPROXIMATION for REAL LINEAR MATERIALS such as for polymers, and also for metals and ceramics at high temperature but low stresses. However for metals and ceramics at high stresses, strain rates are proportional not to the first power of stress, but to a higher power on the order of five.

Rheological Modeling of Steady-State Creep



Strain response of the model

$$\varepsilon = \varepsilon_e + \varepsilon_c = \frac{\sigma'}{E_1} + \varepsilon_c$$

The rate of creep strain is related to the stress by the dashpot constant.

$$\dot{\varepsilon}_c = \frac{d\varepsilon_c}{dt} = \frac{\sigma'}{\eta_1} \quad \Rightarrow$$

When solved for ε_c by integration and combined with ε_e gives the strain-time response

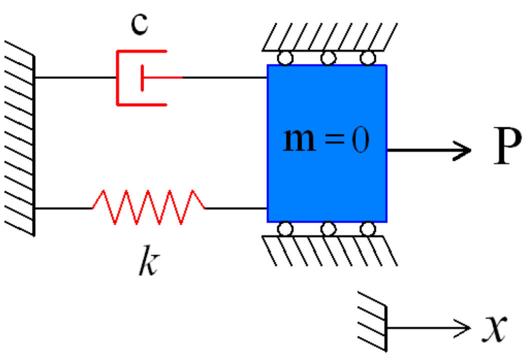
$$\varepsilon = \frac{\sigma'}{E_1} + \varepsilon_c$$

$$\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma' t}{\eta_1}$$

This is the equation of the linear ε-t response during 1-2. After removal of the stress, the elastic strain disappears but the creep strain accumulated during 1-2 remains as permanent strain.

Rheological Modeling of Transient Creep

- Consider simple transient creep model, the elastic spring E_1 is added to the creep strain in the (η_2, E_2) parallel combination. Hence $\varepsilon = \varepsilon_e + \varepsilon_c$ applies. Stress is the sum of the separate stresses in the spring and the dashpot.



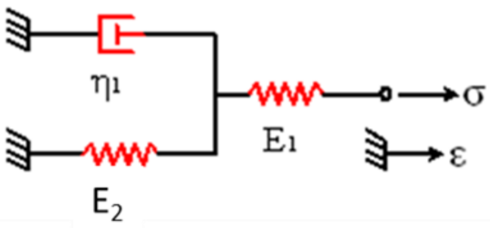
$$\sigma = E_2\varepsilon + \eta_2\dot{\varepsilon} \rightarrow \dot{\varepsilon}_c = \frac{d\varepsilon_c}{dt} = \frac{\sigma - E_2\varepsilon_c}{\eta_2}$$

- Solving this equation for the case of a constant stress σ′ gives ε-t response

$$\varepsilon = \frac{\sigma'}{E_2}\left(1 - e^{-E_2 t/\eta_2}\right)$$

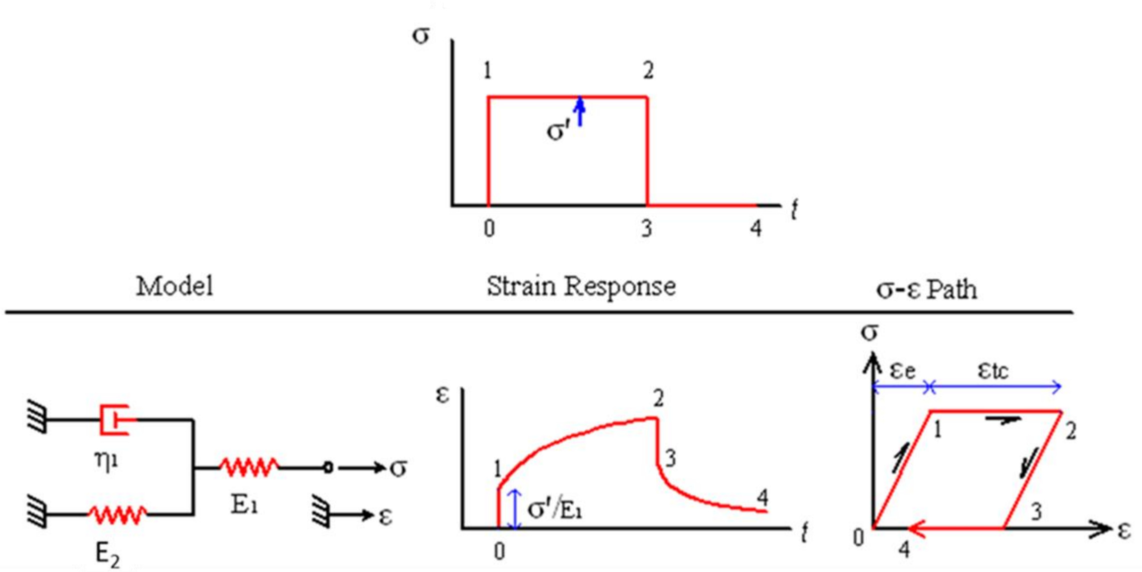
Rheological Modeling of Transient Creep

Finally when elastic strain is added



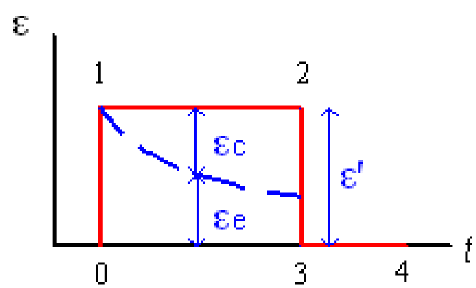
$$\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma'}{E_2}\left(1 - e^{-E_2 t/\eta_2}\right)$$

From this equation it is seen that; ***ε increases and strain rate decreases with time during time interval 1-2***

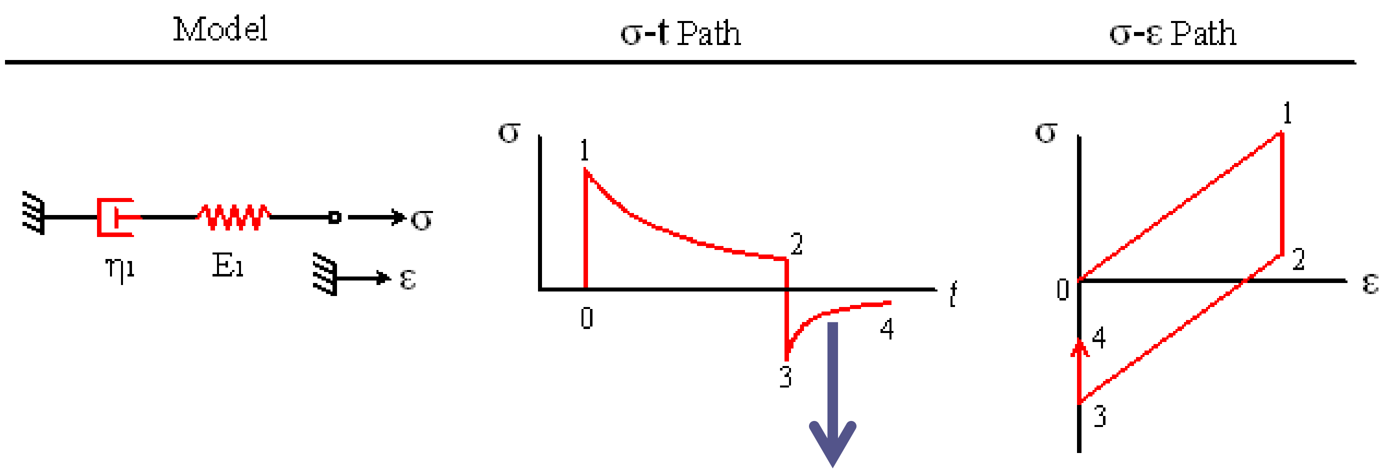


5.2.3. Relaxation Behavior of Elastic, Steady-State Creep Model

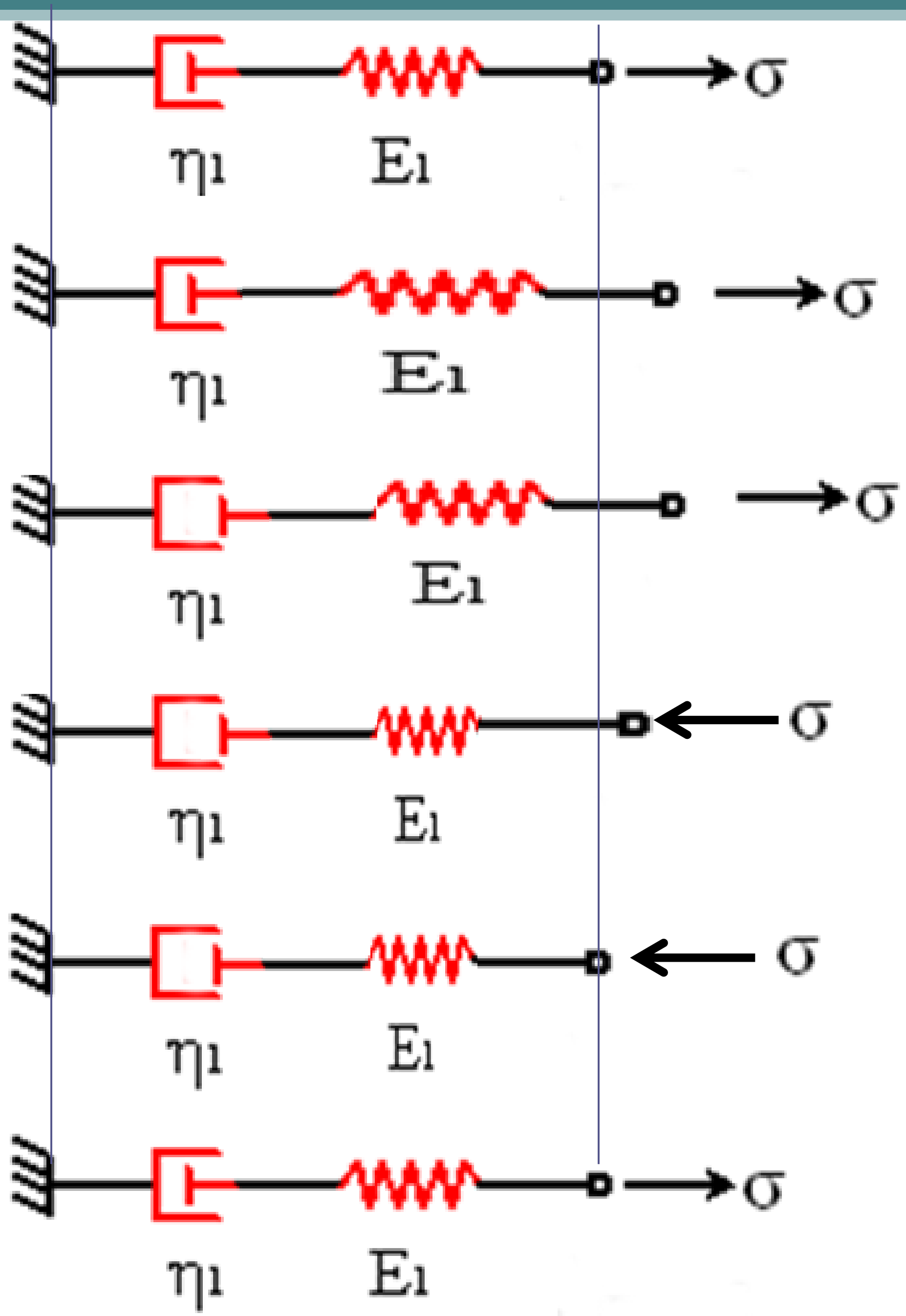
Relaxation; decrease in stress when a material is held at constant strain



With time strain in the dashpot increases and the strain in the spring decreases due to total strain being held constant.



Additional relaxation (if stress is forced into compression)



t=0 $\varepsilon_T = 0$

$\varepsilon_T = \varepsilon_e$ (Step 0 - 1)

$\varepsilon_T = \varepsilon_e + \varepsilon_c$ (Step 1 – 2)

(Step 2-3)

$\varepsilon_T = 0$ (Step 3)

σ approaches 0 - $\varepsilon_T = 0$ (Step 4)

5.2.3. Relaxation Behavior of Elastic, Steady-State Creep Model, cont'd

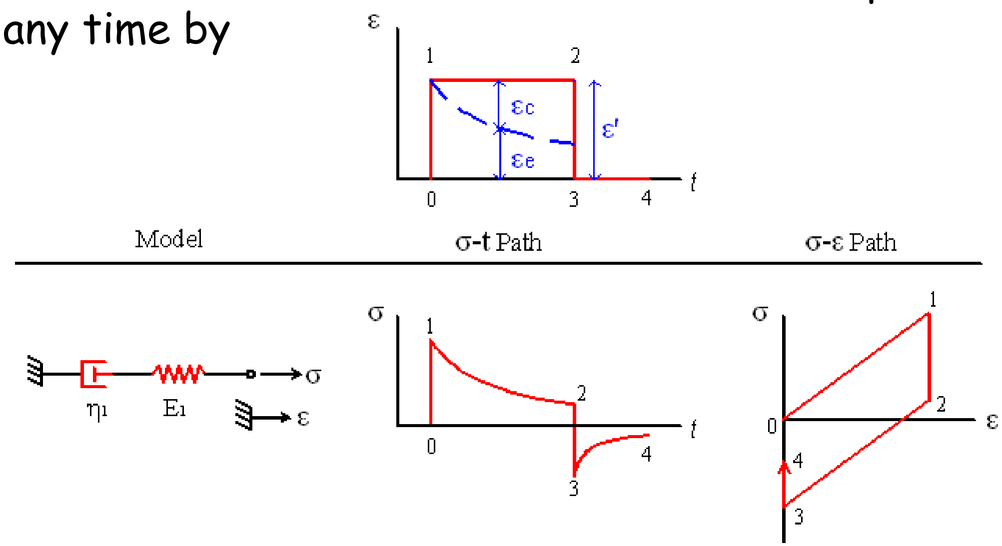
$$\underbrace{\varepsilon'}_{\text{Constant total strain}} = \varepsilon_e + \underbrace{\varepsilon_c}_{\text{Creep strain}}$$

- The stress necessary to maintain the constant strain is related to the elastic strain by;

$$\sigma = E_1 \varepsilon_e$$

- Since ε_e is decreasing, this requires that σ must also decrease. The rate of creep strain is related to the value of σ at any time by

$$\dot{\varepsilon}_c = \frac{d\varepsilon_c}{dt} = \frac{\sigma}{\eta_1}$$



- Combining these two and solving;

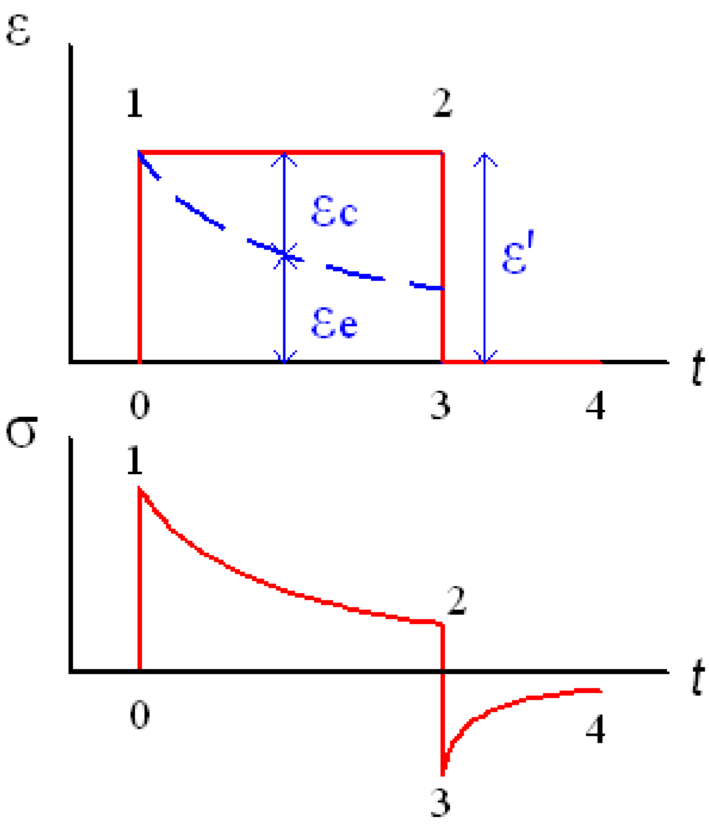
$$\sigma = E_1 \varepsilon' e^{-E_1 t / \eta_1}$$

$$(\sigma - t \ curve \ 1-2)$$

- Relaxation is the same phenomenon as creep, differing only in that it is observed under constant strain rather than constant stress.

Example: $\sigma = E_1 \varepsilon' e^{-E_1 t/\eta_1}$

Derive the equation given above which describes stress relaxation in the elastic, steady state creep model as given by 1-2 of the below figure. Base this on the below equations.



$$\underbrace{\varepsilon' = \varepsilon_e + \varepsilon_c}_{(1)} \qquad \underbrace{\sigma = E_1 \varepsilon_e}_{(2)} \qquad \underbrace{\dot{\varepsilon}_c = \frac{d\varepsilon_c}{dt} = \frac{\sigma}{\eta_1}}_{(3)}$$

Solution:

1) Differentiate both sides of equations (1) and (2) with respect to time, noting that $d\varepsilon'/dt=0$ as ε' is held constant.

$$0 = \dot{\varepsilon}_e + \dot{\varepsilon}_c \qquad \frac{d\sigma}{dt} = \dot{\sigma} = E_1\dot{\varepsilon}_e$$

2) Substitute $d\varepsilon_e/dt$ from 2nd equation and also $d\varepsilon_c/dt$ from equation 3 into the 1st equation to obtain;

$$\frac{1}{E_1}\frac{d\sigma}{dt} + \frac{\sigma}{\eta_1} = 0$$

3) Separate the variables σ and t and integrate both sides of the equation.

$$\frac{d\sigma}{dt} = -\frac{\sigma}{\eta_1}E_1 \Rightarrow \int \frac{d\sigma}{\sigma} = \int -\frac{E_1}{\eta_1}dt$$

Solution cont'd:

$$\ln\sigma = -\frac{E_1}{\eta_1}t + C$$

where C is a constant of integration that can be evaluated by noting that the creep strains is initially zero, so that;

$$\sigma = E_1\varepsilon' \text{ at } t = 0 \qquad \text{then} \quad \ln\sigma = \ln E_1\varepsilon' = C$$

$$\text{This gives} \qquad C = \ln E_1\varepsilon'$$

4) Substituting for C and solving for σ then gives the desired result

$$\ln\sigma = \ln E_1\varepsilon' - \frac{E_1}{\eta_1}t \quad \Rightarrow \quad \ln\left(\frac{\sigma}{E_1\varepsilon'}\right) = -\frac{E_1}{\eta_1}t$$

$$e^{-E_1 t/\eta_1} = \frac{\sigma}{E_1\varepsilon'} \quad \Rightarrow \quad \sigma = E_1\varepsilon' e^{-E_1 t/\eta_1}$$

5.3 Elastic Deformation

Elastic Constants

Homogenous material: A material that has the same properties at all points within the solid.

Isotropic material: If the properties are the same in all directions, the material is isotropic.

Ex: A bar of a homogenous and isotropic material is subject to an axial stress σ_x.

$$\varepsilon_x = \frac{L - L_i}{L_i} = \frac{\Delta L}{L_i}$$

(the strain in the direction of the stress)

$$\underbrace{\varepsilon_y = \frac{d - d_i}{d_i} = \frac{\Delta d}{d_i}}$$

(the strain in any direction perpendicular to the stress)

(-) since the cross-section is reduced.

Q1

- A steel is represented by the elastic, perfectly plastic model with constants E_1=200GPa, and σ_0=400MPa.
  - Plot the stress-strain response for loading to a strain of 0.01. Of this total strain, how much is elastic and how much is plastic?
  - Also plot the response following (a) if the strain is now decreased until it reaches zero.

Q2

- For the elastic linear hardening model, how is the behavior affected by changing E2 while E1 remains constant? You may wish to enhance your discussion by including a sketch showing how the stress-strain path varies with E2.

Q3

- An aluminum alloy is represented by the elastic, linear hardening model with constants E_1=70Gpa, and E_2=2GPa, and σ_0= 350MPa. Plot the stress-strain response for loading to a strain of ε= 0.02. Of this total strain, how much is elastic and how much is plastic?

Q4

- At 600 °C, a silica glass has an elastic modulus of E=60GPa, and a tensile viscosity of η=1000GPa.s. Assuming that the elastic, steady-state creep model applies, determine the response to a stress of 10 MPa maintained for 1 minute and then removed. Plot both strain versus time and stress versus strain for a total time interval of 2 minutes.

Q5

- A polymer is used for shrink-on banding to keep cardboard boxes from popping open during shipment of merchandise. The tension in the banding is observed to have dropped to 90% of its initial value after three months. Estimate how long it will take for the tension to drop to 50% of its original value. The polymer may be assumed to behave according to an elastic, steady state creep model.

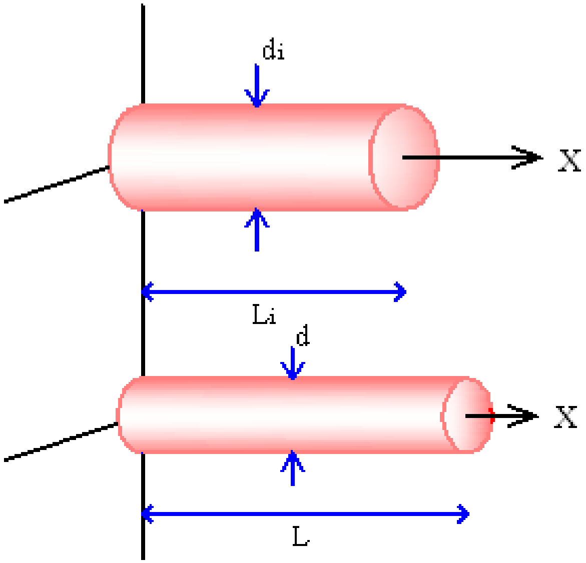
5.3 Elastic Deformation, cont'd

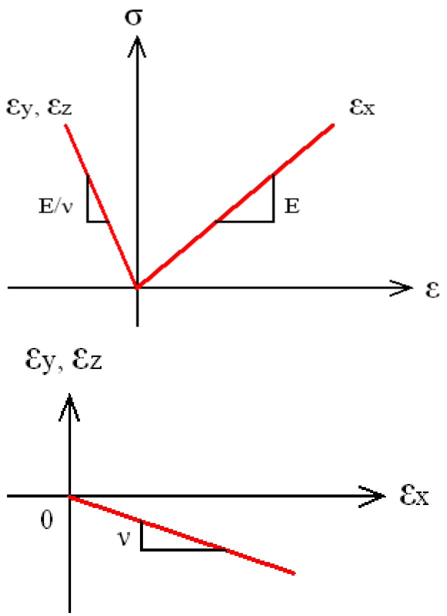
If the bar (material) shows LINEAR ELASTIC behavior, then 2 elastic constants are needed to characterize the material!

$$1)\ Elastic\ Modulus\ \ E = \sigma_x / \varepsilon_x$$

$$2)\ Poisson's\ ratio\ \ \nu = -\frac{transverse\ strain}{longitudinal\ strain} = -\frac{\varepsilon_y}{\varepsilon_x}$$

(Since ε_y is (-), v is (+))





The slope of curve (ε_y vs. ε_x) is (-v)

5.3 Elastic Deformation, cont'd

$$\sigma_x = E\varepsilon_x \quad \left(Substitute\ \nu = -\frac{\varepsilon_y}{\varepsilon_x} \Rightarrow \varepsilon_x = -\frac{\varepsilon_y}{\nu} \right)$$

$$\sigma_x = -E\frac{\varepsilon_y}{\nu}$$

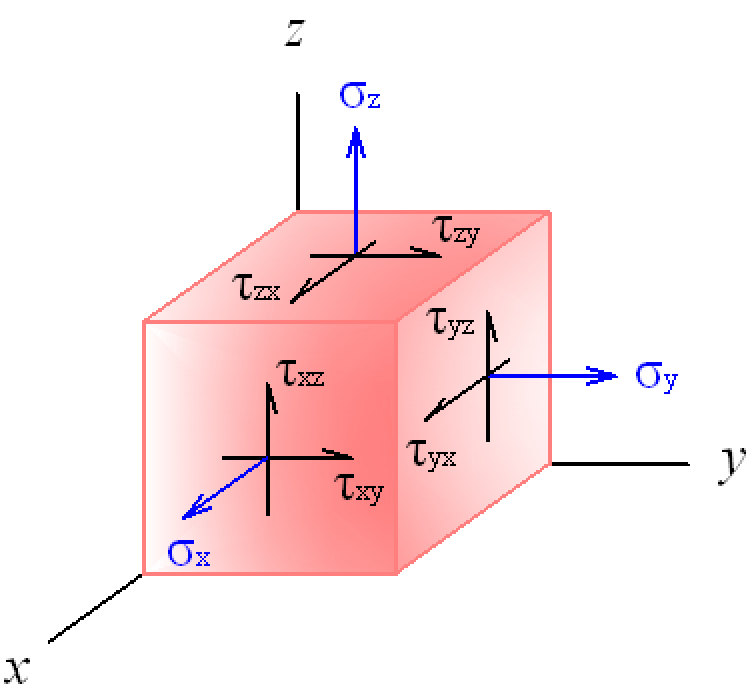
E → widely vary for different materials.

ν → is generally around 0.3 and does not vary outside the range 0 to 0.5 except under very unusual circumstances.

Hooke`s Law for 3 Dimensions

Small strain theory: Most engineering materials is modeled in a way at relatively low stresses below the yield strength beyond which the behavior becomes non-linear and inelastic. Also original dimensions and cross-sectional areas are used in the above discussion to determine stress and strain. This approach is appropriate if dimensional changes are small.

General state of stress at a point;



Normal stresses in 3 directions and shear stresses on 3 planes

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$$

τ_{ab} a → the direction showing the normal of the surface on which the shear stress is acting

b → the direction of shear stress

Hooke`s Law for 3 Dimensions

Consider normal stresses (with small strain assumption)

Stresses in the x direction causes a strain in the x direction $\rightarrow \frac{\sigma_x}{E}$

Same stress (σ_x) also causes strains in the y and z directions which is equivalent to $\rightarrow -\nu \frac{\sigma_x}{E}$

Resulting strain in each direction

$Stress$	x	y	z
σ_x	$\frac{\sigma_x}{E}$	$-\nu \frac{\sigma_x}{E}$	$-\nu \frac{\sigma_x}{E}$
σ_y	$-\nu \frac{\sigma_y}{E}$	$\frac{\sigma_y}{E}$	$-\nu \frac{\sigma_y}{E}$
σ_z	$-\nu \frac{\sigma_z}{E}$	$-\nu \frac{\sigma_z}{E}$	$\frac{\sigma_z}{E}$

Add the columns to obtain strains.

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \end{aligned}\right\}(1)$$

Hooke`s Law for 3 Dimensions, cont'd

Shear Strains

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (2)$$

Note: Shear strains on a given plane is not affected by the shear stresses on other planes.

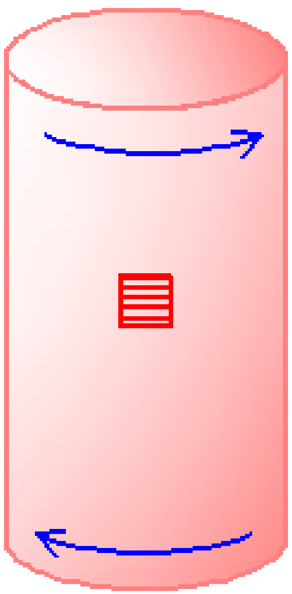
When put together Eqs. (1) and (2) are called Generalized Hooke`s law.

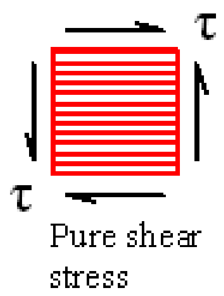
Only 2 elastic constants are needed for an isotropic material so that one of E, G and v can be considered redundant. Anyone of these can be calculated from the other two.

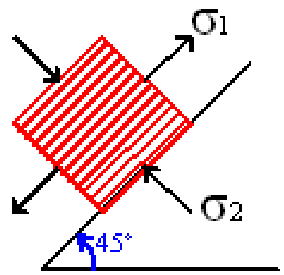
$$G = \frac{E}{2(1+\nu)}$$

Derivation:

Consider a state of pure shear stress as in round bar under torsion.

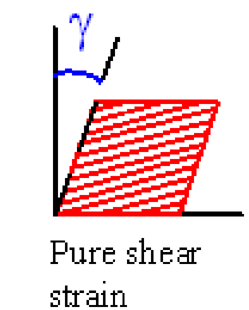


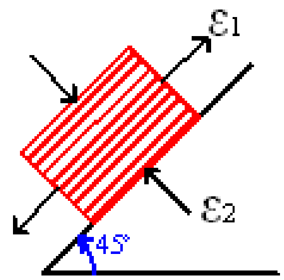




$$\left.\begin{aligned}\sigma_1 &= \tau \\ \sigma_2 &= -\tau \\ \sigma_3 &= 0\end{aligned}\right\}$$

Eq. representation on principal planes





$$\left.\begin{aligned}\varepsilon_1 &= \gamma/2 \\ \varepsilon_2 &= -\gamma/2 \\ \varepsilon_3 &= 0\end{aligned}\right\}$$

Eq. representation on principal planes

Derivation:

State of shear stress, τ, can be equivalently represented by principal normal stresses on planes rotated 45° with respect to the direction of pure shear. (Similarly, the shear strain is equivalent to normal strains as shown also on 45° planes). Let the (x,y,z) directions for the equation (Gen. Hooke`s Law) correspond to the principal (1,2,3) directions. Then;

$$\sigma_x = \tau \qquad \sigma_y = -\tau \qquad \sigma_z = 0 \qquad \varepsilon_x = \gamma/2 \qquad Then;$$

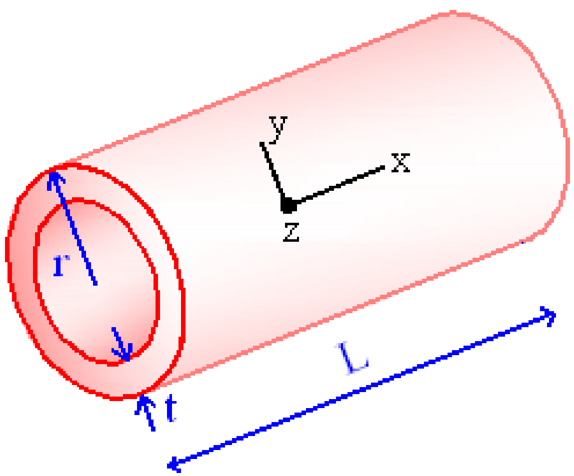
$$\frac{\gamma}{2} = \frac{1}{E}\left(\tau - \nu\left[-\tau + 0\right]\right)$$

$$\gamma = \frac{2(1+\nu)}{E}\tau \rightarrow G = \frac{\tau}{\gamma} \rightarrow G = \frac{E}{2(1+\nu)}$$

Example:

A cylindrical pressure vessel 10 m long has closed ends, a wall thickness of 5 mm, and a diameter at mid-thickness of 3 m. If the vessel is filled with air to a pressure of 2 MPa, how much do the length, diameter and wall thickness change and in each case is the change an increase or a decrease?

The vessel is made of a steel having elastic modulus E=200GPa and Poisson`s ratio v=0.3. Neglect any effects associated with the details of how the ends are attached.



Solution:

r/t is small so that it is reasonable to employ the thin-walled assumption and the resulting stress equations from mechanics of materials.

$$\sigma_x = \frac{pr}{2t} = \frac{(2MPa)(1500mm)}{2(5mm)} = 300MPa$$

$$\sigma_y = \frac{pr}{t} = \frac{(2MPa)(1500mm)}{(5mm)} = 600MPa$$

The value of σ_z varies from (-p) on the inside wall to zero on the outside, and its value for the present case is everywhere sufficiently small that $\sigma_z \approx 0$ can be used. Substitute these stresses and the known E and v into Hooke`s Law.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_x = \frac{1}{200000}\left[300 - 0.3(600)\right] = 6\times10^{-4}$$

$$\varepsilon_y = 2.55\times10^{-3}$$

$$\varepsilon_z = -1.35\times10^{-3}$$

These strains are related to the changes in length ΔL, circumference Δ(πd), diameter Δd, and thickness Δt, as follows.

$$\varepsilon_x = \frac{\Delta L}{L} \quad \varepsilon_y = \frac{\Delta \pi d}{\pi d} = \frac{\Delta d}{d} \quad \varepsilon_z = \frac{\Delta t}{t}$$

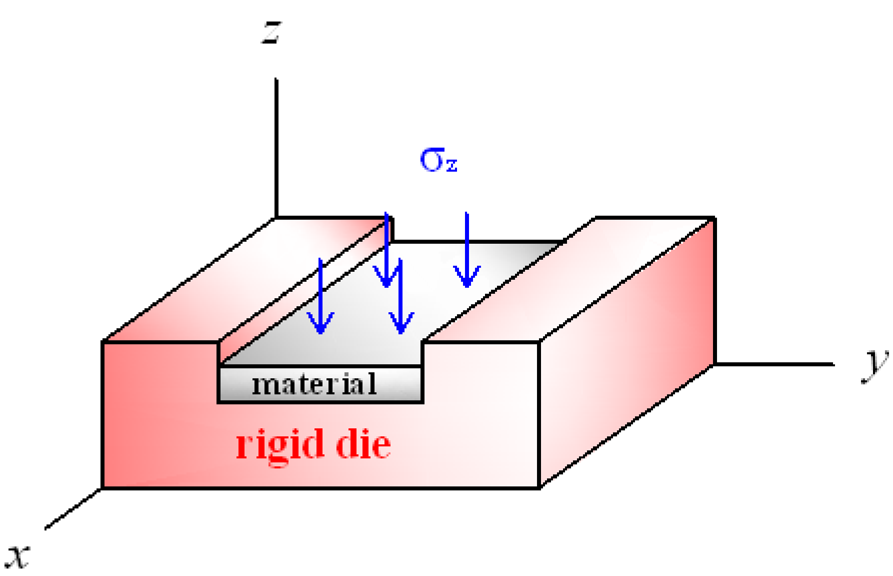
Substituting the strains from above and the known dimensions gives:

$$\Delta L = 6mm \quad \Delta d = 7.65mm \quad \Delta t = -6.75\times10^{-3}mm$$

Example 5.3:

A sample of a material subjected to a compressive stress σ_Z is confined so that it cannot deform in the y-direction but deformation is permitted in the x-direction. Assume that the material is isotropic and exhibits linear-elastic behavior.

Determine the following in terms of σ_Z and the elastic constants of the material:



a) Stress that develops in the y-direction
b) Strain in the z-direction
c) Strain in the x-direction
d) The stiffness $E` = \sigma_Z/\varepsilon_Z$ in the z-direction. Is this apparent modulus equal to the elastic modulus E from a uniaxial test on the material? Why or why not?
e) Assume that the compressive stress in the Z direction has a magnitude of 75 MPa and that the block is made of a copper alloy, and then calculate σ_y, ε_z, ε_x and E′.

Solution:

$$a)\quad \sigma_x = 0 \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left[\sigma_x + \sigma_z\right]\right)$$

$$\varepsilon_y = 0 \qquad 0 = \frac{1}{E}\left(\sigma_y - \nu\sigma_z\right)$$

$$\sigma_z !! \qquad \sigma_y = \nu\sigma_z$$

$$b)\quad \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left[\sigma_x + \sigma_y\right]\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left[0 + \nu\sigma_z\right]\right)$$

$$\varepsilon_z = \frac{1}{E}\left(1 - \nu^2\right)\sigma_z$$

$$c)\quad \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left[\sigma_y + \sigma_z\right]\right)$$

$$\varepsilon_x = \frac{1}{E}\left(0 - \nu\left[\nu\sigma_z + \sigma_z\right]\right)$$

$$\varepsilon_x = -\frac{\nu}{E}\left(1 - \nu\right)\sigma_z$$

$$d)\quad Apparent\ stiffness: E' = \frac{\sigma_z}{\varepsilon_z} = \frac{E}{1 - \nu^2}$$

This value is greater than E such as E`=1.10 E for a typical value of v=0.3. The value E is the ratio of stress to strain only for the uniaxial case and such ratios for any other case are determined by behavior according to the three-D form of Hooke`s law.

e) For the copper alloy E=130 GPa and ν = 0,343. The compression stress requires that a negative sign be applied, so that σ_z = -75MPa. Substituting these quantities into the equations previously derived gives;

$$\sigma_y = \nu\sigma_z = (0,343)(-75MPa) = -25,7MPa$$

$$\varepsilon_z = \frac{1-\nu^2}{E}\sigma_z = \frac{1-0,343^2}{130000MPa}(-75MPa) = -509x10^{-6}$$

$$\varepsilon_x = -\frac{\nu(1+\nu)}{E}\sigma_z = -\frac{0,343(1+0,343)}{130000MPa}(-75MPa) = 266x10^{-6}$$

$$E' = \frac{E}{1-\nu^2} = \frac{130000MPa}{1-0,343^2} = 147300MPa$$

Example 5.17

Strains are measured on the surface of a brass alloy part as follows:

- ε_x: 190 x 10^{-6}
- ε_y: -760 x 10^{-6}
- γ_{xy}: 300 x 10^{-6}

Estimate the in-plane stresses σ_x, σ_y, and τ_{xy} and also the strain ε_z normal to the surface. (Assume that the gages were bonded to the metal when there was no load on the part, that there has been no yielding, and that no loading is applied directly to the surface, so that $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

5.25 A block of isotropic material is stressed in the x- and y-directions, but rigid walls prevent deformation in the z-direction, as shown in Fig. P5.25. The ratio of the two applied stresses is a constant, so that $\sigma_y = \lambda\sigma_x$.

(a) Does a stress develop in the z-direction? If so, obtain an equation for σ_z as a function of σ_x, λ , and the elastic constant ν for the material.

(b) Determine the stiffness $E' = \sigma_x/\varepsilon_x$ for the x-direction as a function of only λ and the elastic constants E and ν for the material.

(c) Compare this apparent modulus E' with the elastic modulus E as obtained from a uniaxial test. (Suggestion: Assume that $\nu = 0.3$ and consider λ values of -1, 0, and 1.)

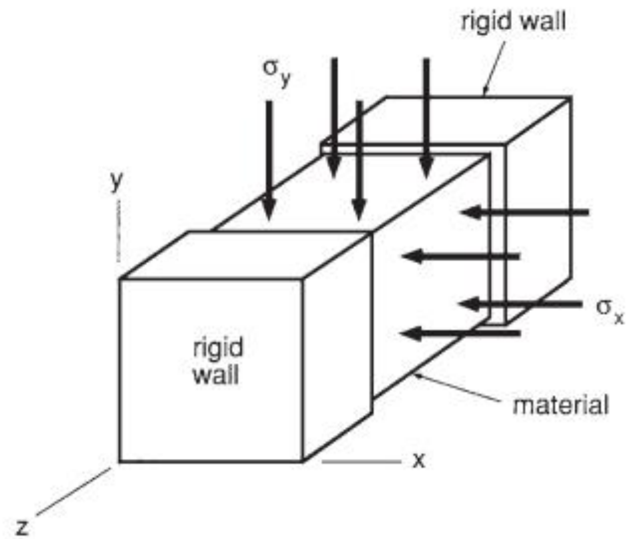
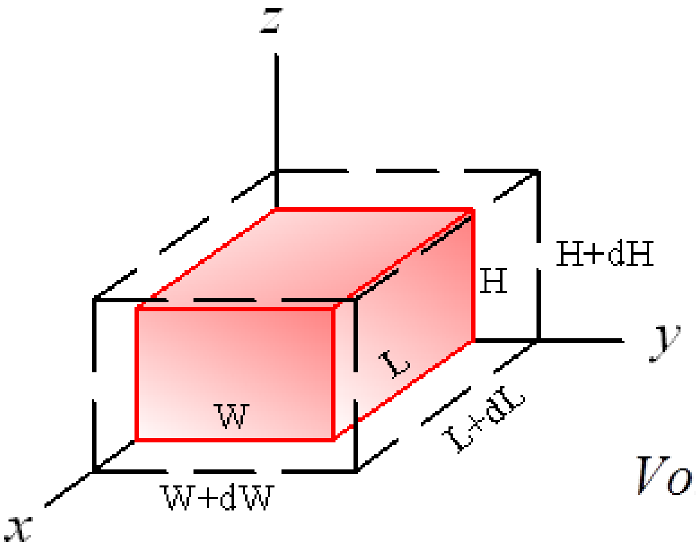


Figure P5.25

5.3.3. Volumetric strain and hydrostatic stress



Volume change due to normal strains
(Shear strains are not involved since they cause no volume change only distortion)

$$V = L\,W\,H$$

$$Volume\ change : dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial W} dW + \frac{\partial V}{\partial H} dH$$

Evaluating the partial derivatives and dividing both sides by V=LWH gives;

$$\underbrace{\frac{dV}{V}}_{\substack{Volumetric \\ Strain\,(or\ dilitation) \\ \varepsilon_v}} = \frac{dL}{L} + \frac{dW}{W} + \frac{dH}{H} \qquad \frac{dV}{V} = \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

For an isotropic material (Use Generalized Hooke`s Law)

$$\varepsilon_v = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

[if v=0.5 then ε_v=0 even if there are non zero stresses]

Average normal stress is called hydrostatic stress.

$$\sigma_h = \frac{\left(\sigma_x + \sigma_y + \sigma_z\right)}{3} \quad \Rightarrow \varepsilon_v = \frac{3(1-2\nu)}{E}\sigma_h$$

Volumetric strain is proportional to the hydrostatic stress.

The constant of proportionality relating these is called bulk modulus;

$$B = \frac{\sigma_h}{\varepsilon_v} = \frac{E}{3(1-2\nu)}$$

ε_v and σ_h are called invariant quantities. This means that they will always have the same values regardless of the choice of coordinate system. In other words a different choice of x, y, z axes will cause the various stress and strain components to have different values, but the sum of them will have the same value for any coordinate system.

Thermal strains

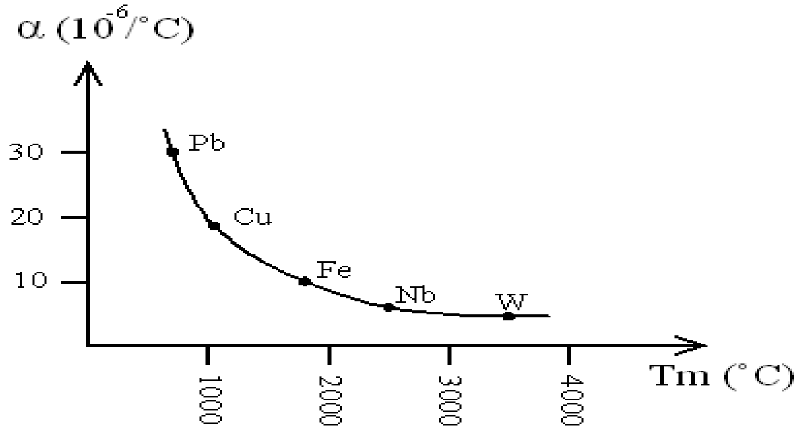
- This is a special class of elastic strain that results from expansion with increasing temperature or contraction with decreasing temperature.
- Increased temperature → increased vibration of atoms (increased inter-atomic spacing)
- In isotropic materials, the effect is the same in all directions. Over a limited range of temperatures, the thermal strains at a given temperature T can be assumed to be proportional to the temperature change, ΔT.

$$\varepsilon = \alpha\left(T - T_c\right) = \alpha\left(\Delta T\right)$$

T_o : Reference temperature where the strains are taken to be zero.
α : Coefficient of thermal expansion (1/°C)

Thermal strains

Thermal strains (and therefore values of α) are <u>smaller</u> where the atomic bonding is stronger.



$$\alpha \downarrow \text{ when } T_m \uparrow$$

In an isotropic material, since uniform thermal strains occur in all directions, Hooke's law for 3 dimensions can be generalized to include thermal effects.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha\left(\Delta T\right)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha\left(\Delta T\right)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha\left(\Delta T\right)$$

If free thermal expansion is prevented by geometric constraints, then a sufficient ΔT will cause large stresses that are of engineering significance to develop.

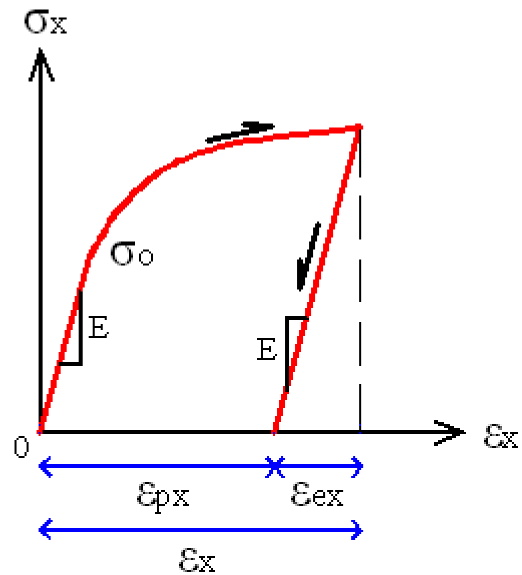
5.3.5. Comparison with plastic and creep deformations

Plastic Deformation

Stress-Strain Behavior

- ✓ Time independent deformation that is not recovered after unloading is termed as plastic deformation.

Typical behavior for metals



$$\varepsilon_{ex} = Elastic\,strain$$

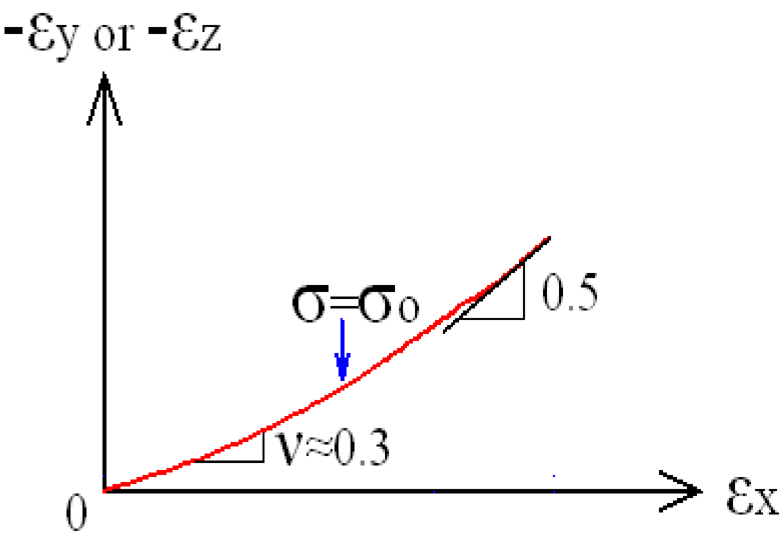
$$\varepsilon_{px} = Plastic\,strain$$

$$\varepsilon_x = \varepsilon_{ex} + \varepsilon_{px}$$

$$\varepsilon_x = \frac{\sigma_x}{E} + \varepsilon_{px}$$

5.3.5. Comparison with plastic and creep deformations

Plastic Deformation



- ✓ Slope deviates from elastic Poisson`s ratio and approaches 0.5 following yielding. Prior to yielding, the slope is simply ν.
- ✓ Elastic strain in any transverse direction is still related to that in the x-direction by the elastic value of Poisson`s ratio.

$$\varepsilon_{ey} = \varepsilon_{ez} = -\nu\varepsilon_{ex}$$

- ✓ Constant analogous to Poisson`s ratio has a value near 0.5.

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_{ex} - 0.5\varepsilon_{px}$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon_v = \underbrace{\varepsilon_{ex} + \varepsilon_{px}}_{\varepsilon_x} + \underbrace{\left(-\nu\varepsilon_{ex}\right) - 0.5\varepsilon_{px}}_{\varepsilon_y} + \underbrace{\left(-\nu\varepsilon_{ex}\right) - 0.5\varepsilon_{px}}_{\varepsilon_z}$$

$$\varepsilon_v = \varepsilon_{ex}\left(1 - 2\nu\right)$$

5.3.5. Comparison with plastic and creep deformations

Plastic Deformation

For the general case of a three-dimensional state of stress where plastic deformation occur, Hooke`s law in the form of Eqs. (1) gives only the elastic strains (ε_{ex}, ε_{ey}, ε_{ez}). A second set of Eqs. are used for the plastic strains.

$$\left.\begin{aligned}
\varepsilon_{px} &= \frac{1}{E_p}\left(\sigma_x - 0.5\left[\sigma_y + \sigma_z\right]\right)\\
\varepsilon_{py} &= \frac{1}{E_p}\left(\sigma_y - 0.5\left[\sigma_x + \sigma_z\right]\right)\\
\varepsilon_{pz} &= \frac{1}{E_p}\left(\sigma_z - 0.5\left[\sigma_x + \sigma_y\right]\right)
\end{aligned}\right\} Eqs\,(3)$$

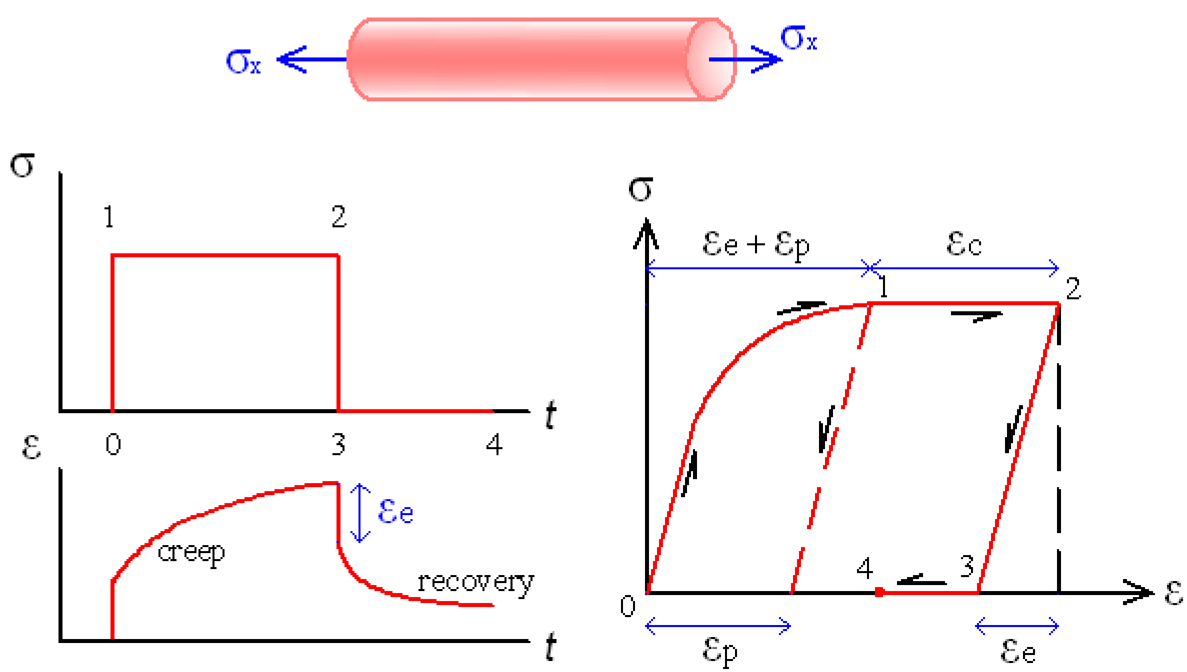
→ Constant E is replaced by E_p

→ Poisson`s ratio replaced by 0.5

5.3.5. Comparison with plastic and creep deformations

Creep Deformation

Significant time-dependent deformations occur in engineering metals and ceramics at elevated temperatures. They also occur at room temperature in low-melting temperature metals, such as lead and in many other materials, such as glass, polymers and concrete.



Stress-time step applied to a material exhibiting strain response that includes elastic, plastic and creep components.

5.3.5. Comparison with plastic and creep deformations

Creep Deformation

- ✓ A stress is rapidly applied to a sample of material.
- ✓ The instantaneous deformation is a combination of elastic and plastic strain.
- ✓ Plastic portion could be isolated by immediate unloading (dashed line).
- ✓ If the stress is instead maintained, creep deformation ε_c may occur that is a combination of transient creep and steady-state creep.
- ✓ Removal of the stress causes the elastic strain ε_e to be instantly recovered. Some of the creep strain may be recovered after waiting a period of time as indicated by 3-4.

Unrecovered strain = part of the creep strain + all of plastic strain

5.3.5. Comparison with plastic and creep deformations

Creep Deformation

- ✓ Examples from real materials; creep strains as large as 100% in flexible vinyl (plasticized PVC) may be mostly recovered after unloading. Viscous flow in glass is similar to steady-state creep rheological model and is not recovered upon unloading.
- ✓ Equations analogous to Hooke`s law except the rates of creep strain are obtained.

$$\dot{\varepsilon}_{cx} = \frac{1}{\eta}\left(\sigma_x - 0.5\left[\sigma_y + \sigma_z\right]\right)$$

$$\dot{\varepsilon}_{cy} = \frac{1}{\eta}\left(\sigma_y - 0.5\left[\sigma_x + \sigma_z\right]\right)$$

$$\dot{\varepsilon}_{cz} = \frac{1}{\eta}\left(\sigma_z - 0.5\left[\sigma_x + \sigma_y\right]\right)$$

- ✓ In the simplest application η is a constant and the creep behavior is of the steady state type.
- ✓ In more complex applications η can be a stress-dependent variable. Replacing Poisson`s ratio by 0.5 specifies that creep strain does not contribute to volume change.

5.4. Anisotropic materials

- Real materials are never perfectly isotropic. In some cases, the differences in properties for different directions are so large that analysis assuming isotropic behavior is no longer a reasonable approximation.
- Due to the presence of stiff fibers in particular directions, composite materials can be highly anisotropic, and engineering design and analysis for these materials requires the use of a more general version of Hooke's law then was presented above.

5.4.1. Anisotropic Hooke's law

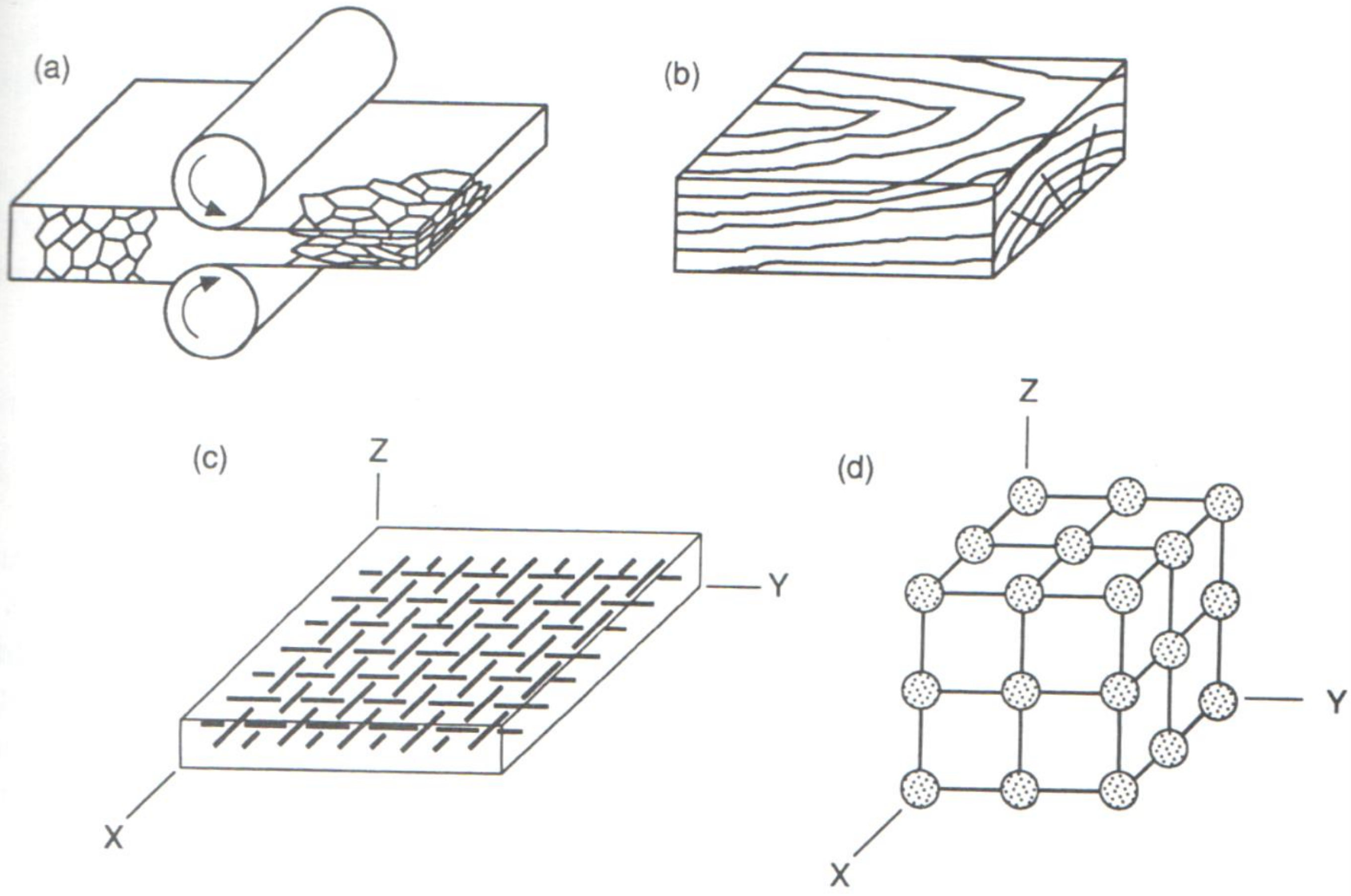


Figure 4.16 Anisotropic materials: (a) rolled metal, (b) wood, (c) glass-fiber cloth in an epoxy matrix, and (d) a crystal with cubic unit cell. (Adapted from [Crandall 59] p. 224; used with permission; copyright ©1959 by McGraw-Hill Pub. Co.)

5.4.1. Anisotropic Hooke's law, cont'd

General anisotropic form of Hooke's law..

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Materials constants. These constants change if the orientation of the x-y-z coordinate system is changed.

5.4.1. Anisotropic Hooke's law, cont'd

Properties of the matrix

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Each unique S_{ij} has a different nonzero value. The matrix is symmetrical about its diagonal in such a way that there are two occurences of each S_{ij}, where $i \neq j$, so that there are 21 independent constants..

5.4.1. Anisotropic Hooke's law, cont'd

In the isotropic case, the constants do not depend on the orientation of the coordinate axes, and most of the constants are either zero or have the same values as other ones.

$$[S_{ij}] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix}$$

How many independent constants are there?

5.4.2. Orthotropic and cubic materials

If the material possesses symmetry about three orthogonal planes, that is, about planes oriented 90° to each other, then a special case called an orthotropic material exist. In this case, Hooke's law has a form of intermediate complexity between the isotropic and the general anisotropic cases.

Constants for the orthotrophic case

$$[S_{ij}] = \begin{bmatrix} \frac{1}{E_X} & -\frac{\nu_{YX}}{E_Y} & -\frac{\nu_{ZX}}{E_Z} & 0 & 0 & 0 \\ -\frac{\nu_{XY}}{E_X} & \frac{1}{E_Y} & -\frac{\nu_{ZY}}{E_Z} & 0 & 0 & 0 \\ -\frac{\nu_{XZ}}{E_X} & -\frac{\nu_{YZ}}{E_Y} & \frac{1}{E_Z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{YZ}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{ZX}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{XY}} \end{bmatrix}$$

To deal with the situation of the S_{ij} values changing with the orientation of the x-y-z coordinate system, it is convenient to define the values for the directions parallel to the planes of symmetry in the material.

Examples include an orthorombic single crystal, where $\alpha = \beta = \gamma = 90°$, but $a \neq b \neq c$, and fibrous composite materials with fibers in directions such that there are three orthogonal planes of symmetry.

Constants for the orthotrophic case

Properties of the matrix

In the matrix there are three moduli Ex, Ey, Ez for three different directions in the material. There are also three different shear moduli Gxy, Gyz, and Gzx corresponding to three planes.

$$\upsilon_{ij} = -\frac{\varepsilon_j}{\varepsilon_i}$$

this gives the transverse strain in the j –direction due to a stress in the i - direction.

Due to the symmetry of Sij diagonal, following holds:

$$\frac{\upsilon_{ij}}{E_i} = \frac{\upsilon_{ji}}{E_j}$$

where $i \neq j$ and i, j = X, Y or Z. These relationships reduce the number of independent Poisson's ratios to three for a total of nine independent constants. These constants only apply for a Special X-Y-Z coordinate system.

Constants for the orthotrophic case

Properties of the matrix – cubic case

If the material has the same properties in the X, Y, and Z directions, then it is called a cubic material. In this case, all three E_i have the same value E_x, all three G_{ij} have the same value Gxy, and all six Poisson's ratios have the same value υ_{xy}.

Then, how many independent constants are there?

Examples include all single crystals with a cubic structure, such as BCC, FCC, and diamond cubic crystals.

5.4.3. Fibrous composites

Many applications of composite materials involve thin sheets or plates that have symmetry corresponding to the orthotropic case, such as simple unidirectional or woven arrangements of fibers.

For plates or sheets, stresses that do not lie in the X-Y plane of the sheet are usually small, so that plane stress with $\sigma_Z = \tau_{YZ} = \tau_{ZX} = 0$ is a reasonable assumption. Although strains ε_Z still occur, these are not of particular interest, so that Hooke's law is used in a reduced form.

5.4.3. Fibrous composites, cont'd

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_X} & -\frac{\nu_{YX}}{E_Y} & 0 \\ -\frac{\nu_{XY}}{E_X} & \frac{1}{E_Y} & 0 \\ 0 & 0 & \frac{1}{G_{XY}} \end{bmatrix} \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix}$$

Capital letters indicate that the stresses, strains, and elastic constants are expressed only for directions parallel to the planes of symmetry of the material.

Remember $\frac{\upsilon_{YX}}{E_Y} = \frac{\nu_{XY}}{E_X}$ Then;

How many independent constants are there?

5.4.3. Fibrous composites, cont'd

TABLE 4.3 ELASTIC CONSTANTS AND DENSITY FOR FIBER-REINFORCED EPOXY WITH 60% UNIDIRECTIONAL FIBERS BY VOLUME

Reinforcement			Composite				
Type	E_r	ν_r	E_X	E_Y	G_{XY}	ν_{XY}	ρ
	GPa (10^3 ksi)		GPa (10^3 ksi)				g/cm^3
E-glass	72.3 (10.5)	0.22	45 (6.5)	12 (1.7)	4.4 (0.64)	0.25	1.94
Kevlar 49	124 (18.0)	0.35	76 (11.0)	5.5 (0.8)	2.1 (0.3)	0.34	1.30
Graphite (T-300)	218 (31.6)	0.20	132 (19.2)	10.3 (1.5)	6.5 (0.95)	0.25	1.47
Graphite (GY-70)	531 (77.0)	0.20	320 (46.4)	5.5 (0.8)	4.1 (0.6)	0.25	1.61

Note: For approximate matrix properties, use $E_m = 3.5$ GPa (510 ksi) and $\nu_m = 0.33$.

Sources: Data in [ASM 87] pp. 175-178, and [Kelly 89] p. 262.

Example 5.4

A plate of the epoxy reinforced with unidirectional Kevlar 49 fibers in Table below is subject to stresses as follows: σ_X=400, σ_Y=12, and τ_{XY}=15 MPa, where the coordinate system is as given below on the right hand side of the slide . Determine the in-plane strains ε_X, ε_Y, and γ_{XY}.

TABLE 4.3 ELASTIC CONSTANTS AND DENSITY FOR FIBER-REINFORCED EPOXY WITH 60% UNIDIRECTIONAL FIBERS BY VOLUME

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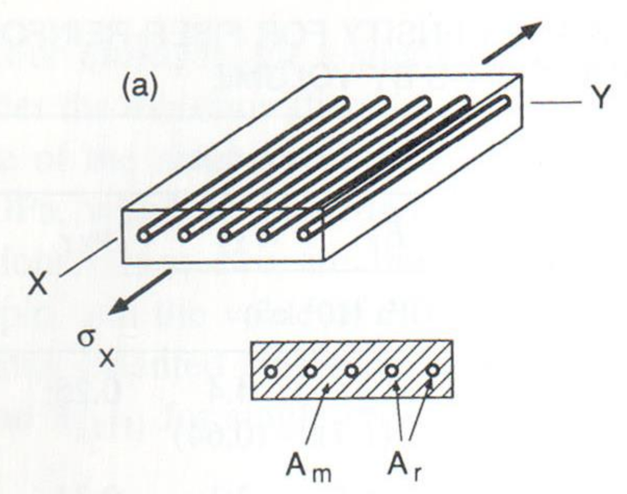
Note: For approximate matrix properties, use E_m = 3.5 GPa (510 ksi) and ν_m = 0.33.

Sources: Data in [ASM 87] pp. 175-178, and [Kelly 89] p. 262.



Solution:

$$\varepsilon_X = \frac{\sigma_X}{E_X} - \frac{\nu_{YX}}{E_Y}\sigma_Y \qquad \varepsilon_Y = \frac{\sigma_Y}{E_Y} - \frac{\nu_{XY}}{E_X}\sigma_X \qquad \gamma_{XY} = \frac{\tau_{XY}}{G_{XY}}$$

Since v_{yx} is not given in the table, employ the below relation:

$$\frac{\nu_{XY}}{E_X} = \frac{\nu_{YX}}{E_Y} = \frac{0.34}{76000} = 4.474\times10^{-6}\ 1/MPa$$

$$\varepsilon_X = \frac{400}{76000} - \left(4.474\times10^{-6}\right)\left(12\right) = 0.00521$$

$$\varepsilon_Y = \frac{12}{5500} - \left(4.474\times10^{-6}\right)\left(400\right) = 0.00039$$

$$\gamma_{XY} = \frac{15}{2100} = 0.00714$$

5.4.4. Elastic modulus parallel to fibers

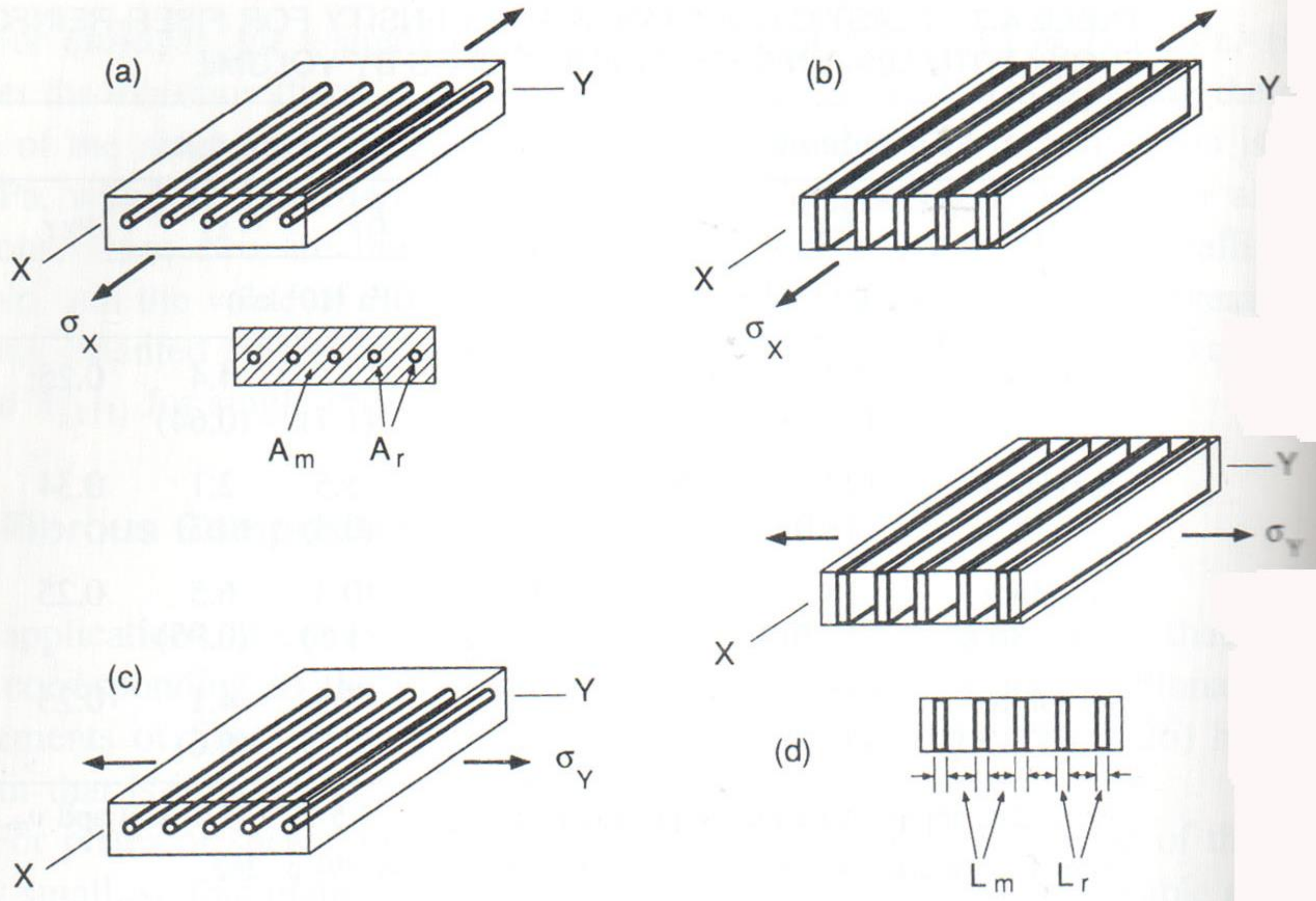


Figure 4.17 Composite materials with various combinations of stress direction and unidirectional reinforcement. In (a) the stress is parallel to fibers, and in (b) to sheets of reinforcement, whereas in (c) and (d) the stresses are normal to similar reinforcement.

5.4.4. Elastic modulus parallel to fibers

<u>*Case 1:*</u>

<u>*Assumptions:*</u>

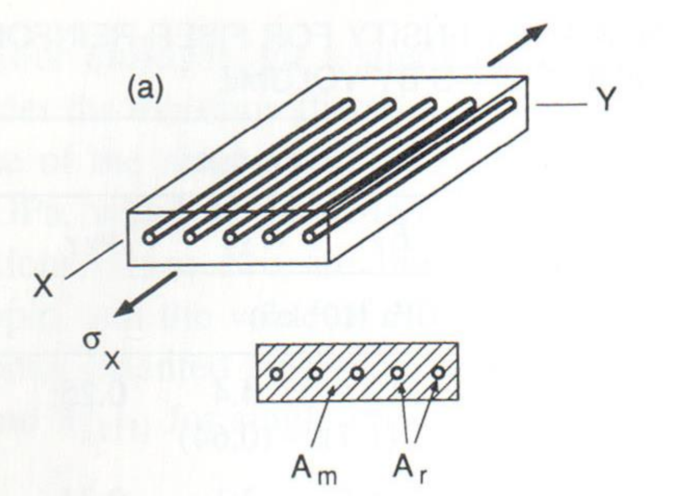
1) Fibers be an isotropic material with elastic constants E_r, υ_r and σ_r.
2) Matrix is another isotropic material, E_m, υ_m and σ_m.
3) Fibers are perfectly bound to the matrix, so that fibers and matrix deform as a unit, resulting in the same strain ε_X in both.

- ✓ Areas occupied by matrix and fibers A_r, A_m respectively

$$A = A_r + A_m \rightarrow Total\,area$$

- ✓ Applied force must be the sum of contributions from fibers and matrix

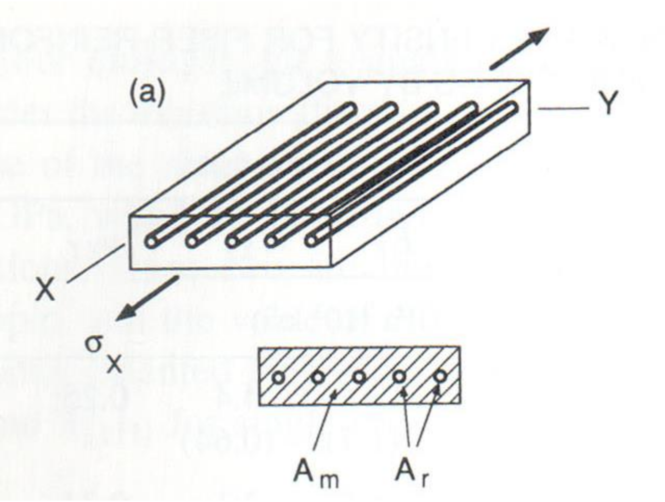
$$\sigma_X A = \sigma_r A_r + \sigma_m A_m$$



5.4.4. Elastic modulus parallel to fibers

From the definitions;

$$\left.\begin{aligned}\sigma_X &= E_X\varepsilon_X\\ \sigma_r &= E_r\varepsilon_r\\ \sigma_m &= E_m\varepsilon_m\end{aligned}\right\} \qquad \varepsilon_X = \varepsilon_r = \varepsilon_m$$



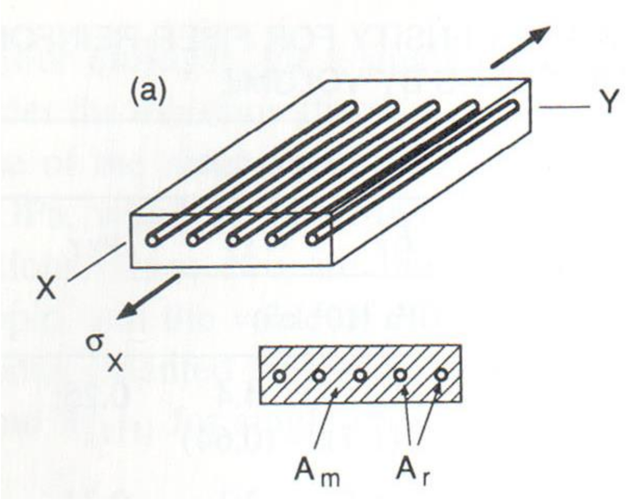
$$E_X\varepsilon_X A = E_r\varepsilon_r A_r + E_m\varepsilon_m A_m$$

$$E_X\varepsilon_X A = \left(E_r A_r + E_m A_m\right)\varepsilon_{r\,\mathrm{or}\,m}$$

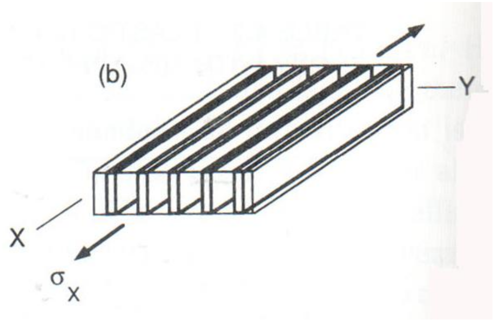
$$E_X = \frac{E_r A_r + E_m A_m}{A}$$

5.4.4. Elastic modulus parallel to fibers

$$\text{Notice that } \left.\begin{aligned} V_r &= \frac{A_r}{A} \\ V_m &= \frac{A_m}{A} \end{aligned}\right\} \text{are the volume fractions of fiber and matrix} \quad (1)$$



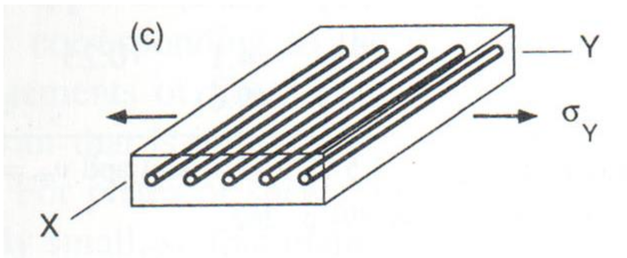
$$\text{Then; } E_X = V_r E_r + V_m E_m$$



This relationship is also valid for Fig 4.17 (b)

5.4.5. Elastic Modulus Transverse to Fibers

Case 3: (4.17 (c))

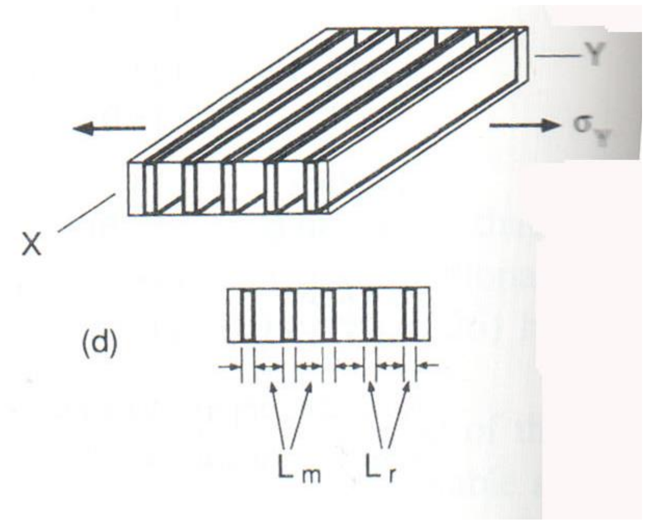


- ✓ Uniaxial loading in the other orthogonal plane direction.
- ✓ Analysis of this case is difficult, therefore analysis of Case 4 (fig d) will be done.
- ✓ E_Y obtained from the analysis of Case 4 is shown to provide a lower bound on the correct value for Case 3 Fig (c).

$$\sigma_Y = \sigma_r = \sigma_m \quad (must\,be\,equal)$$

$$\sigma_Y = E_Y \varepsilon_Y \qquad \sigma_r = E_r \varepsilon_r \qquad \sigma_m = E_m \varepsilon_m$$

Total length in the y direction ➡ $L = L_r + L_m$



5.4.5. Elastic Modulus Transverse to Fibers

- ✓ The changes in these lengths give the strain in the overall composite material and in the reinforcement and matrix portions.

$$\varepsilon_Y = \frac{\Delta L}{L} \qquad \varepsilon_r = \frac{\Delta L_r}{L_r} \qquad \varepsilon_m = \frac{\Delta L_m}{L_m}$$

$$\Delta L = \Delta L_r + \Delta L_m$$

$$\downarrow$$

$$\varepsilon_Y L = \varepsilon_r L_r + \varepsilon_m L_m$$

- ✓ Substitute for the strains (***remember stresses are equal***)

$$\frac{\sigma_Y}{E_Y} L = \frac{\sigma_r}{E_r} L_r + \frac{\sigma_m}{E_m} L_m$$

$$\frac{1}{E_Y} = \frac{1}{E_r}\frac{L_r}{L} + \frac{1}{E_m}\frac{L_m}{L}$$

5.4.5. Elastic Modulus Transverse to Fibers

The length ratios are equivalent to volume fractions.

$$V_r = \frac{L_r}{L} \qquad V_m = 1 - V_r = \frac{L_m}{L} \qquad \textit{to give} \qquad \frac{1}{E_Y} = \frac{V_r}{E_r} + \frac{V_m}{E_m}$$

$$\textit{Solve for } E_Y \quad \Rightarrow \quad E_Y = \frac{E_r E_m}{V_r E_m + V_m E_r}$$

5.4.6. Other Elastic Constants and Discussion

- ✓ Similar logic leads to a estimate of υ_{xy}, the larger of two Poisson`s ratio and also an estimate of the shear modulus.

$$\nu_{XY} = V_r \nu_r + V_m \nu_m$$

$$G_{XY} = \frac{G_r G_m}{V_r G_m + V_m G_r}$$

- ✓ In a laminate, if equal numbers of fibers occur in several directions, such as the 0°, 90°, +45° and -45° directions, the elastic constants may be approximately the same for any direction in the X-Y plane, but different in the Z-direction. Such a material is said to be transversely isotropic.

Examples

1) For the epoxy reinforced with 60 % unidirectional E- glass fibers in Table (slide No:), use the matrix and reinforcement properties given to estimate the composite properties E_X, E_Y, G_{XY}, υ_{XY} and υ_{XY}. How well do your estimates compare with the tabulated values? Can you suggest reasons for any discrepancies? (5.33)
2) A composite material is to be made of tungsten wire aligned in a single direction in an aluminum alloy matrix. The elastic modulus parallel to fibers must be at least 225 GPa, and the elastic modulus perpendicular to the fibers must be at least 100 GPa. (5.43)
   1) What is the smallest volume fraction of wire that can be used?
   2) For the volume fraction of wire chosen in (a), what is the shear modulus of the composite material?

Chapter 6- Complex and Principal States of Stress and Strain

- ✓ Various types of loads may be present on components of machines, vehicles and structures (tension, compression, bending, torsion, pressure etc…)
- ✓ As a result of this, complex states of normal and shear stresses occur that vary in magnitude and direction with location in the component.

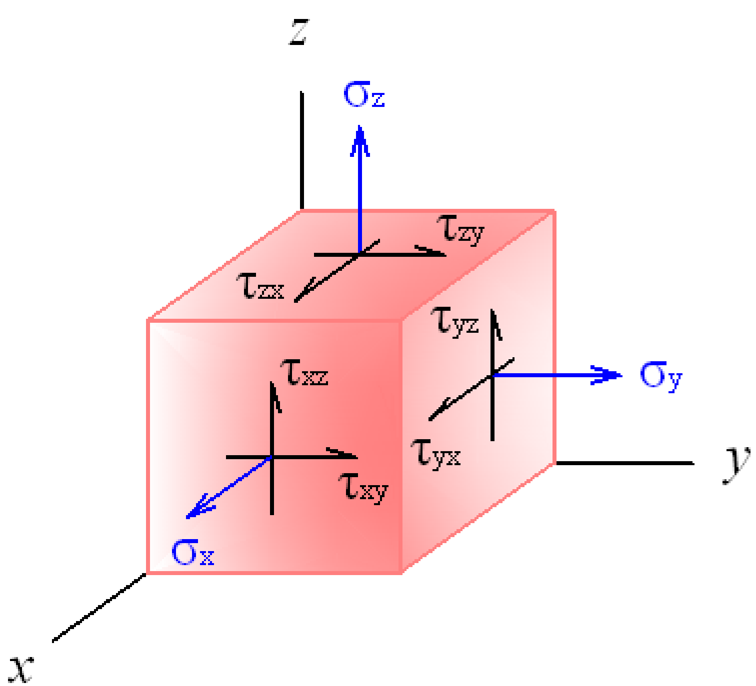
→ Designer must know the locations where the stresses are most severe. Further analysis must be done at these locations.

→ Magnitudes of stress vary with direction and are highest in certain directions.

The highest stresses at a given location are called the ***PRINCIPAL STRESSES*** and the particular directions in which these act are called ***PRINCIPAL AXES.***

6.2 Plane Stress
(Stresses acting on one of the orthogonal planes are "0")

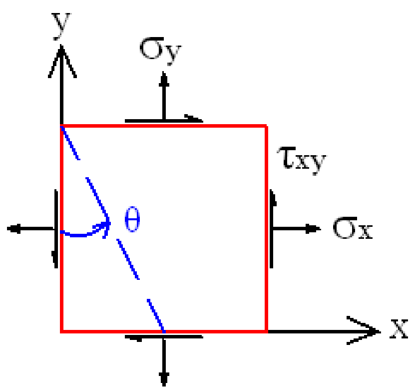
- ✓ Occurs at any free surface in a component.



$$\sigma_z = \tau_{zx} = \tau_{yz} = 0$$

- ✓ Equations of forces requires that; moments must sum to zero about both the x and y axes, requiring in turn that components of τ_{xz} and τ_{zy} acting on the other two planes must also be zero.

Hence the components remaining are σ_x, σ_y and τ_{xy} as shown on a square element of a material.

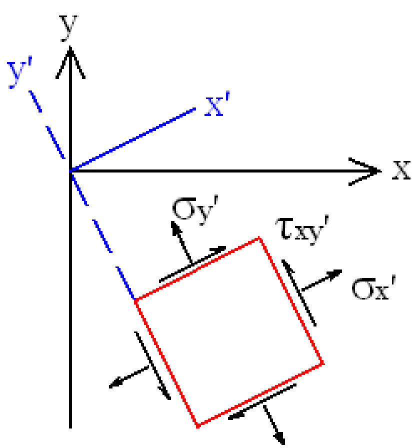


Cubic element viewed parallel to the z-axis.

Positive Directions

1) Tensile normal stresses are positive
2) τ is (+) if arrows on the positive facing sides of the element are in the directions of the positive x-y coordinate axes.

6.2.1 Rotation of Coordinate Axes



- ✓ Value of stress components change to σ_x' σ_y' τ_{xy}' in the new system.
- ✓ This is an equivalent representation of the original ones.
- ✓ Eq. of forces in the x and y directions provides two eq.s.

Summing forces in the x and y directions.

$$\overset{\rightarrow +}{\Sigma}_x = \sigma\cos\theta - \tau\sin\theta - \sigma_x\cos\theta - \tau_{xy}\sin\theta = 0$$

$$\overset{+\uparrow}{\Sigma}_y = \sigma\sin\theta + \tau\cos\theta - \sigma_y\sin\theta - \tau_{xy}\cos\theta = 0$$

Solve for σ *and* τ;

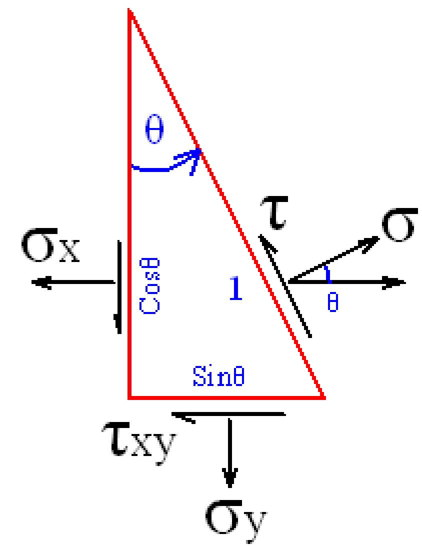
$$\sigma'_x = \sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

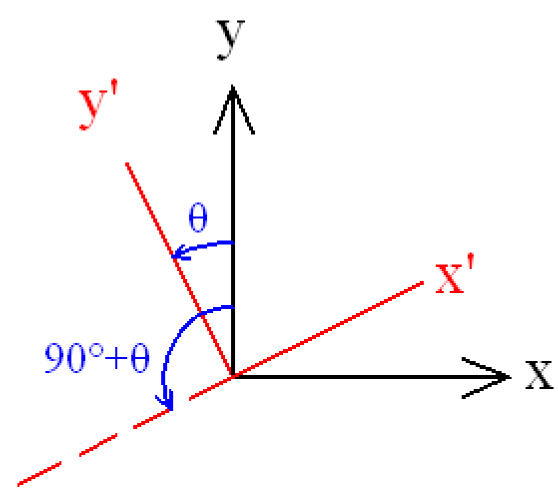
$$\tau'_{xy} = \tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Substitute $\theta + 90^\circ$ *to obtain* σ'_y

θ (counter clockwise → +)

The process is called transformation of axes, so that the equations are called transformation equations.





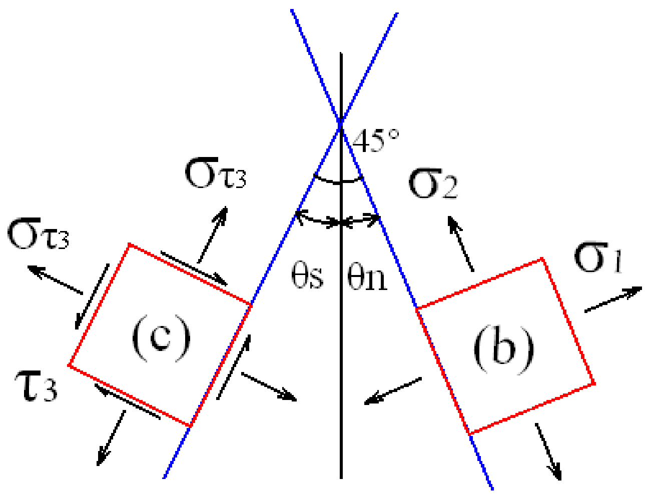
6.2.2 Principal Stresses

Above equations give the variation of σ and τ with directions in the material. Max and min values of σ and τ are of special interest and can be calculated.

1) Take derivative dσ/dθ of $\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

2) Equate the result to "0" to obtain coordinate axes rotations for the minimum and maximum values of σ.

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



2 angles θ_n separated by 90° satisfying this relation. Corresponding stresses (max and min) are principal normal stresses

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Shear stress at orientation θ_n is "0".

6.2.2 Principal Stresses, cont'd

On the planes where principal normal stresses occur shear stress is "0". If shear stress is "0" then the normal stresses are principal normal stresses.

$d\tau/d\theta = 0$ gives the angle (or coordinate axes rotation for the maximum shear stress).

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Corresponding shear stress is

$$\tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

This is the maximum shear stress in the x-y plane and is called the principal shear stress.

Also 2 orthogonal planes where this shear stress occurs are found to have the same normal stresses of

$$\sigma_{\tau_3} = \frac{\sigma_x + \sigma_y}{2}$$

Shows that this is the normal stress accompanying τ_3.

$2\theta_n$ and $2\theta_s$ differ by 90°. Then they are limited to range of ±90°, then one of these must be negative that is clockwise (CW) and we can write;

$$\left|\theta_n - \theta_s\right| = 45^\circ \qquad \tau_3 = \frac{\left|\sigma_1 - \sigma_2\right|}{2} \qquad \sigma_{\tau_3} = \frac{\sigma_1 + \sigma_2}{2}$$

Example 6.1

At a point of interest on the free surface on an engineering component, the stresses with respect to convenient coordinate system in the plane of the surface are σ_x=95, σ_y=25 and τ_{xy}=20 MPa. Determine the principal stresses and the orientations of the principal planes.

1) Find the angle to the coordinate axes for PNS.

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{4}{7} \qquad \theta_n = 14.9^\circ$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1, \sigma_2 = 60 \pm 40.3 \quad \Rightarrow \quad \sigma_1 = 100.3MPa \qquad \sigma_2 = 19.7MPa$$

Alternatively, a more rigorous procedure is to use θ = θn = 14.9° which gives (see transformation eqs. below)

$$\sigma = \sigma'_x = \sigma_1 = 100.3 MPa \qquad (Eq. 6.4)$$

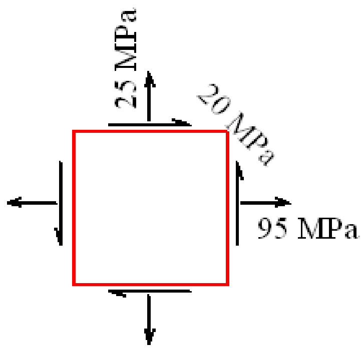
Use of θ = θn + 90°=104.9° then gives the normal stress in the other orthogonal direction.

$$\sigma = \sigma'_y = \sigma_2 = 19.7 MPa$$

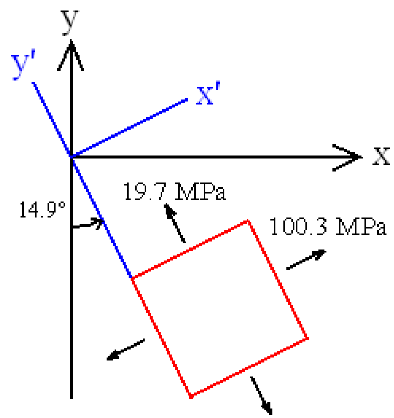
The "0" value of τ at θ = θn can also be verified by using Eq. 6.5.

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad Eq.\ 6.4$$

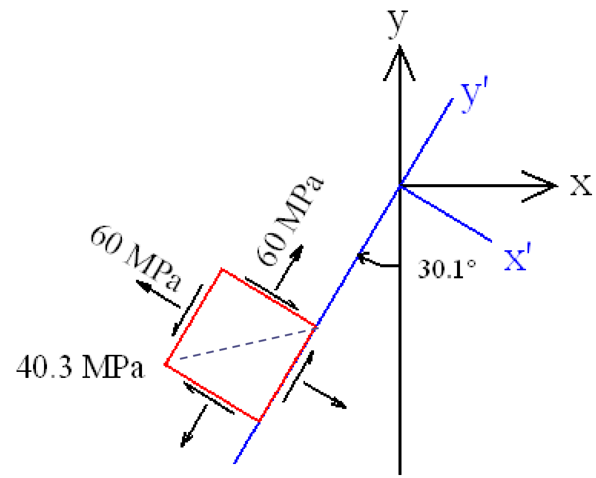
$$\tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad Eq.\ 6.5$$



(a)



Principal normal stress



Principal shear stress

For the equivalent representation where the max. shear stress in the x-y plane occurs,

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = -\frac{7}{4} \qquad \theta_s = -30.1^\circ \ (CW)$$

The stresses for this rotation of the coordinate system may be obtained from σ_1 and σ_2 as calculated above

$$\tau_3 = \frac{|\sigma_1 - \sigma_2|}{2} = \pm 40.3MPa$$

$$\sigma_{\tau_3} = \frac{\sigma_1 + \sigma_2}{2} = 60MPa$$

(Positive shear diagonal (dashed line) must be aligned with the larger of σ_1 and σ_2)

Another method is to use $\theta = \theta s = -30.1°$ which gives $\tau_3 = 40.3$ Mpa.

6.2.3 Mohr's Circle

Developed by Otto Mohr in the 1880`s.

1) Isolate all terms containing 2θ on 1 side.

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad Eq.(a)$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad Eq.(b)$$

2) Then square both sides Eqs. (a) and (b). Sum the result and eliminate θ to obtain

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad which\, has\, the\, below\, form$$

$$(\sigma - a)^2 + (\tau - b)^2 = r^2$$

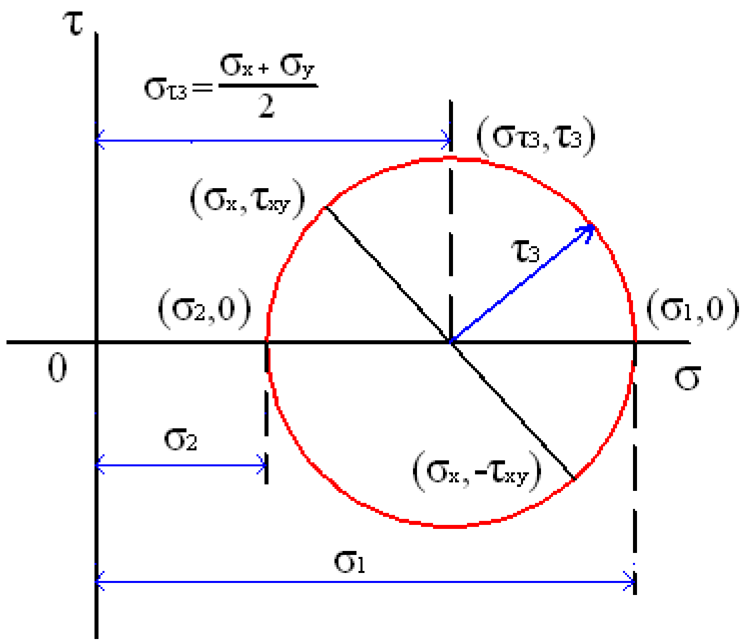
This is the equation of a plot of σ vs. τ with center at coordinates (a, b) and radius r, where;

$$a = \frac{\sigma_x + \sigma_y}{2} \qquad b = 0 \qquad r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Remember; $\sigma_{\tau_3} = \dfrac{\sigma_x + \sigma_y}{2}$ *and* $\tau_3 = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Then; $a = \sigma_{\tau_3}$ *and* $r = \tau_3$

6.2.3 Mohr's Circle, cont'd



Radius τ_3 → max shear stress in the x-y plane
Max and min normal stresses occur along the σ axis.

$$\sigma_1, \sigma_2 = a \pm r = \frac{\sigma_x + \sigma_y}{2} \pm \tau_3$$

Example 6.2:

Repeat the previous example using Mohr`s circle.

σ_x=95MPa, σ_y=25MPa and τ_{xy}=20MPa

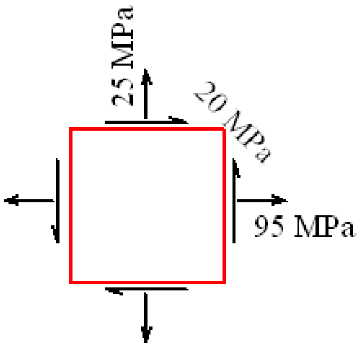
Solution:

The circle is obtained by plotting 2 points that lie at opposite ends of a diameter.

$$(\sigma,\tau)=(\sigma_x,-\tau_{xy})=(95,-20)\,MPa$$

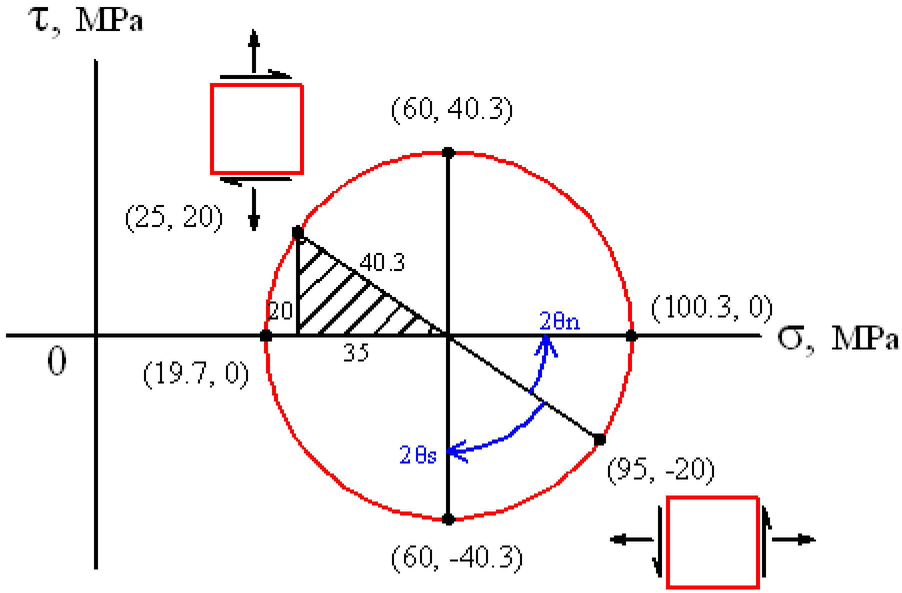
$$(\sigma,\tau)=(\sigma_y,\tau_{xy})=(25,20)\,MPa$$

A negative sign is applied to τ_{xy} for the point associated with σ_x because the shear arrows on the same planes as σ_x tend to cause a CCW rotation. Similarly a positive sign is used for τ_{xy} when associated with σy, due to the CW rotation. The center of the circle must lie on the σ axis at a point halfway between σ_x and σ_y, that is at;



$$(\sigma,\tau)_{center} = \left(\frac{\sigma_x + \sigma_y}{2}, 0\right) = (60,0)\,MPa$$

Solution cont'd:



Shaded triangle has a base of 20 MPa and an altitude of 35 MPa. Then; hypotenuse is the radius of the circle and is also the principal shear stress.

$$\mathit{The\ angle\ with\ the\ }\sigma-\mathit{axis\ is}$$

$$\tan 2\theta_n = 20/35 \qquad \Rightarrow \qquad 2\theta_n = 29.74^\circ\left(CCW\right)$$

A counter clockwise rotation of the diameter of the circle by this diameter gives the horizontal diameter that corresponds to the PNS. Their values are obtained from the center location and radius of the circle.

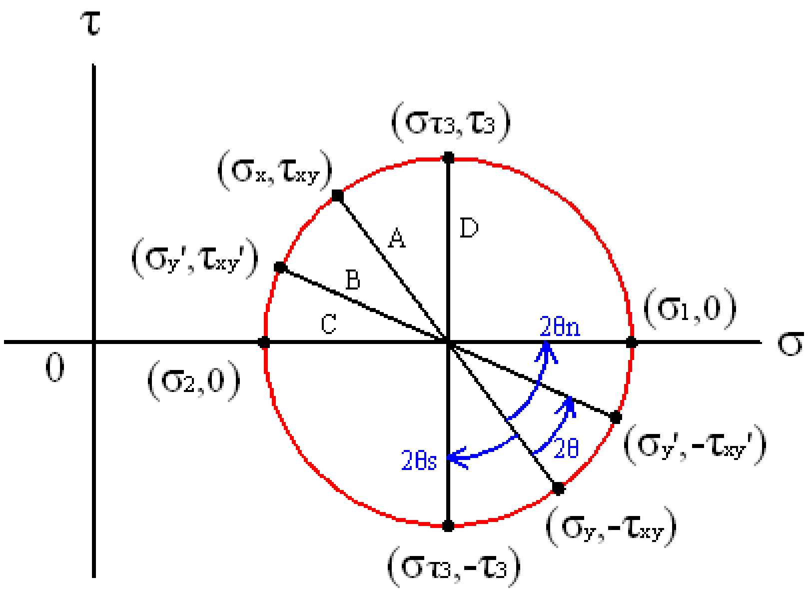
$$\sigma_1,\sigma_2 = \frac{\sigma_x+\sigma_y}{2} \pm \tau_3 = 60 \pm 40.3 \quad \Rightarrow \quad \sigma_1,\sigma_2 = \left(100.3, 19.7\right)MPa$$

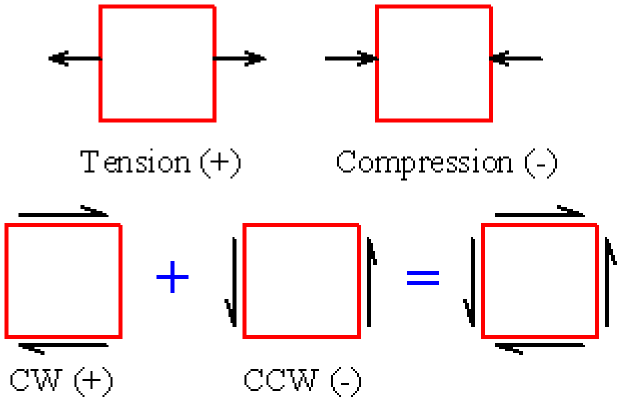
Solution cont'd:

The diameter corresponding to the original state of stress must be rotated CW to obtain the equivalent representation that contains the principal shear stress. Since this is 90° from the σ-axis, the angle of rotation is

$$2\theta_s = 90^\circ - 2\theta_n = 60.26^\circ \ (CW)$$

$$so\,that \ \ \theta_s = 30.1^\circ \ (CW)$$





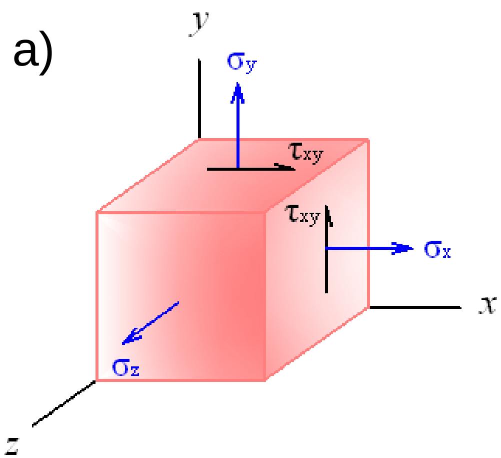
Convention to Distinguish Signs of Shear Stresses

1) Consider complete shear stress to be split into 2 portions.
2) Portion that causes CW rotation is considered (+)
3) Portion that causes CCW rotation is considered (-)
4) For normal stresses, tension is positive and compression (-)
5) 2 Ends of a diameter of the circle could be used to represent the stresses on orthogonal planes in the material.
6) Normal + shear stresses that occur together on one orthogonal plane provide the coordinate point for one end of the diameter and those for the other plane give the opposite end.
7) A rotation of 2θ of this diameter on the circle corresponds to state of stress for a coordinate axis rotation of θ in the same direction in the material. (Diameter B)
8) Rotate by an angle $2\theta n$ (principal normal stresses are obtained)
9) Rotate by an angle $2\theta s$ (principal shear stresses are obtained)

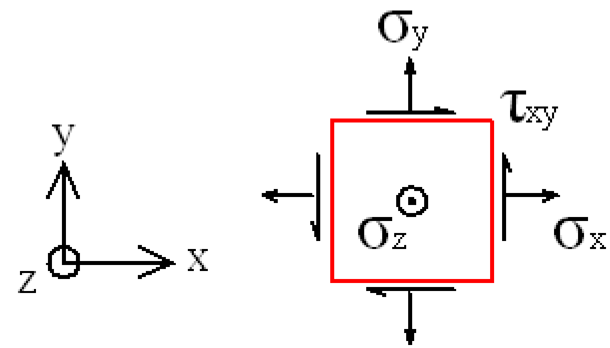
6.2.4 Generalized Plane Stress

$\tau_{yz} = \tau_{xz} = 0$ What is the difference when compared to plane stress ($\sigma z \neq 0$)

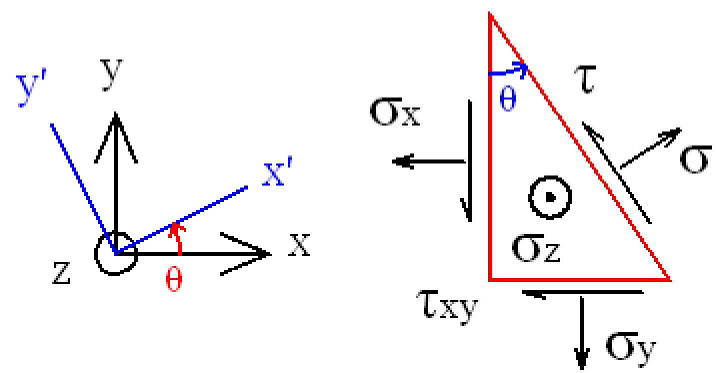
a)



b)



c)



Only difference (from the plane stress) is that σ_z has a nonzero value.

6.2.4 Generalized Plane Stress, cont'd

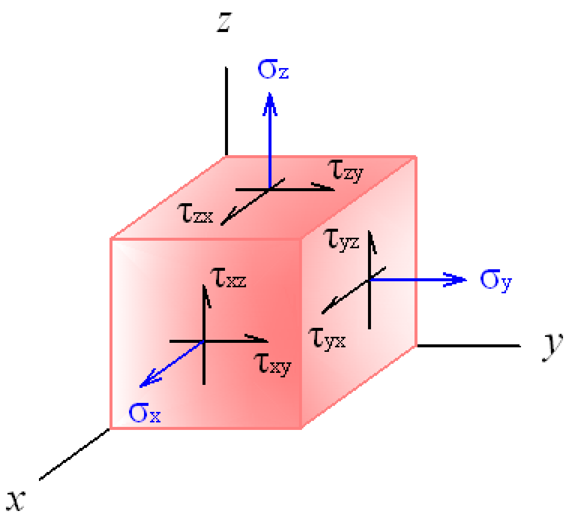
- ✓ The eq.s of equilibrium in the x-y plane with the plane stress are the same!
- ✓ This includes the equations for PNS in the x-y plane (σ_1 and σ_2), and also those for the PSS and the accompanying normal stress (τ_3 and $\sigma_{\tau 3}$).
- ✓ It is simply necessary to note that σ_z remains unchanged for all rotations of the coordinate axes in the x-y plane.
- ✓ Moreover, since Mohr`s circle was also derived from the same equations it can be employed for the x-y plane.

This state of stress is called

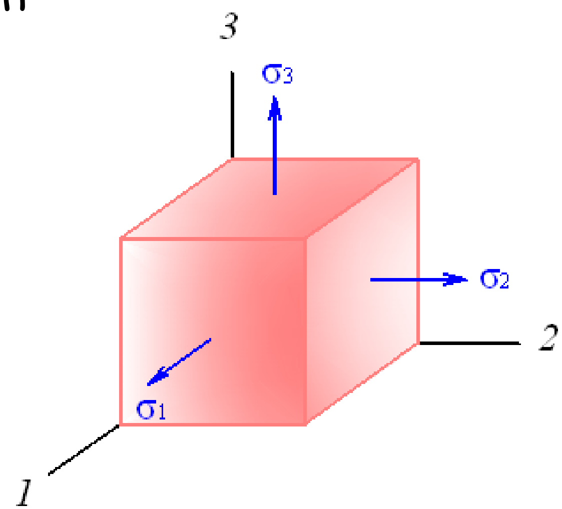
Generalized Plane Stress

6.3 Principal Stresses and The Maximum Shear Stress

Any state of stress on a xyz system



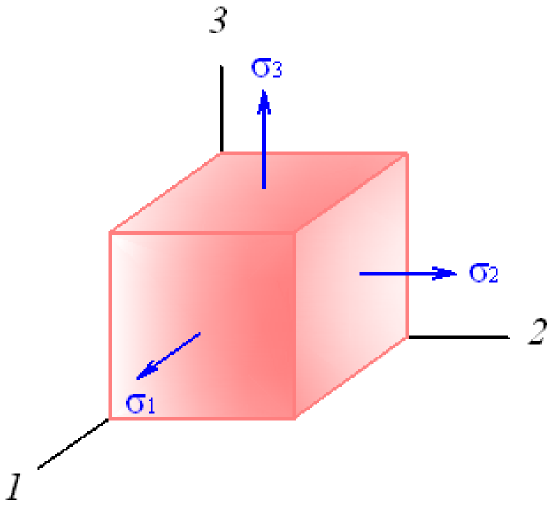
Equivalent representation on a new coordinate system of principal axes 1,2,3 where no shear stresses are present



$$\sigma_1, \sigma_2, \sigma_3 \;\rightarrow\; PNS$$

One is the maximum normal stress
One is the minimum normal stress
The remaining one has an intermediate value

6.3 Principal Stresses and The Maximum Shear Stress, cont'd



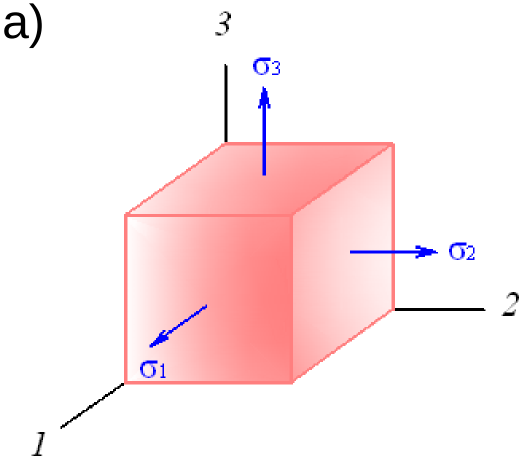
For plane stress or generalized plane stress σ_1 σ_2 and their directions could be found from the previously given equations.

For the general 3-D case?

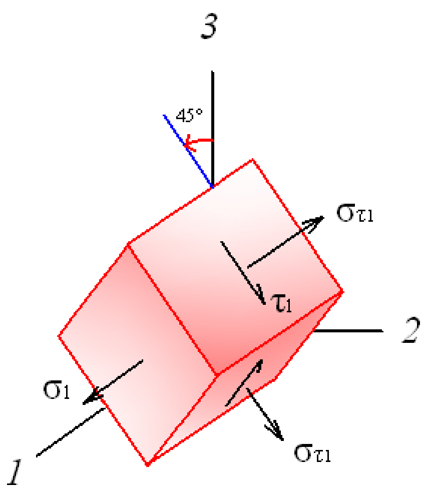
For plane stress condition, directions of PNS (1-2 axes) could be find by θ_n rotation of x-y axes. For the general 3-D case, the 1-2-3 axes are unique directions that may all differ from the original x-y-z directions. General procedure for finding σ_1, σ_2, σ_3 will be given later.

6.3.1 Principal Shear Stresses and The Maximum Shear Stress

a)

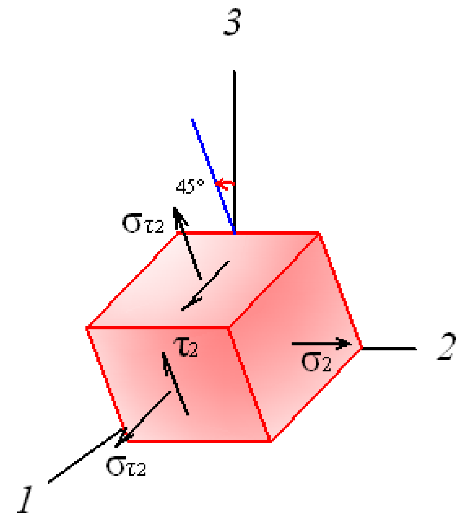


b)



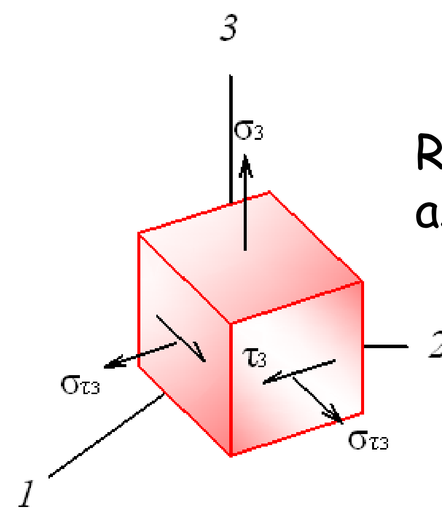
Rotation about the axis of σ_1

c)



Rotation about the axis of σ_2

d)



Rotation about the axis of σ_3

For the figure on the previous slide if equivalent state of stress is found for a 45° rotation about any of the 1,2,3 axes, a shear stress is encountered that is the largest for any rotation about that axis.

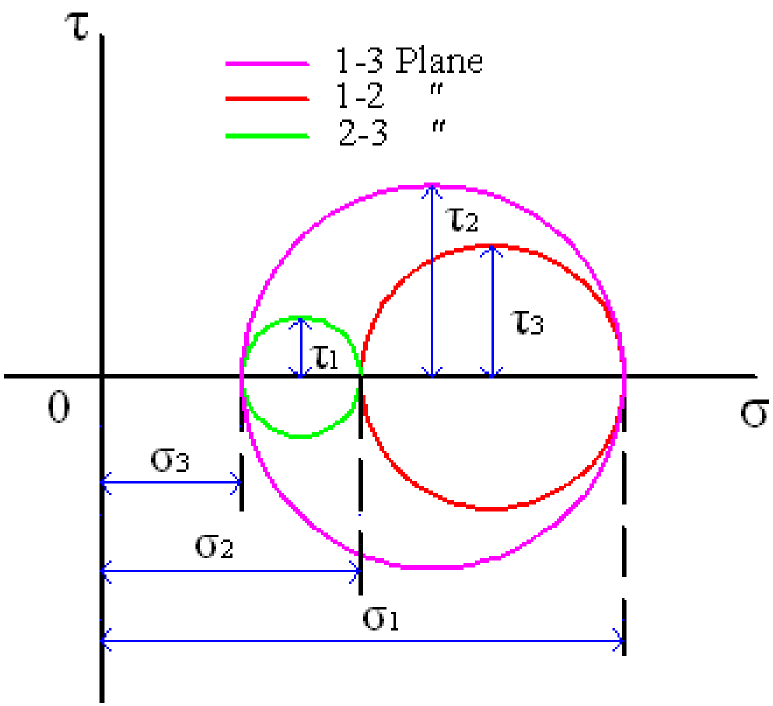
τ_1, τ_2, τ_3 → PSS accompanied by normal stresses $\sigma_{\tau1}$, $\sigma_{\tau2}$, $\sigma_{\tau3}$ that are the same on the 2 shear planes.

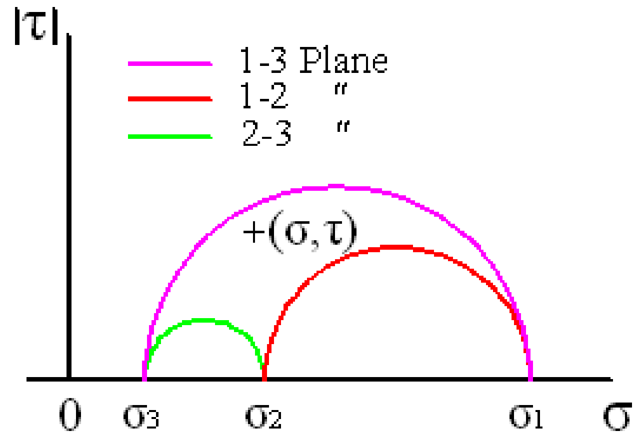
$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} \qquad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} \qquad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{\tau_1} = \frac{\sigma_2 + \sigma_3}{2} \qquad \sigma_{\tau_2} = \frac{\sigma_1 + \sigma_3}{2} \qquad \sigma_{\tau_3} = \frac{\sigma_1 + \sigma_2}{2}$$

$$\tau_{\max} = Max(\tau_1, \tau_2, \tau_3) \qquad \tau_{\max} = \left(\begin{array}{c} \text{the largest of the 3} \\ \text{principal shear stresses.} \end{array} \right)$$

Mohr Circle Applies





Each circle is tangent along the σ axis to the other two.
The radii of the circles are PSSs τ_1 τ_2 τ_3
The centers are located along the σ-axis at the points given by the 3 σ_{Ti} values.
Each plane (where one of the PSSs occurs) is seen to be 45° rotation away from the corresponding planes of PNS.

Example 6.3

For the following state of stress, determine the principal normal stresses, the principal axes and the principal shear stresses:

$\sigma_x = 100 \quad \sigma_y = -60 \quad \sigma_z = 40$ MPa

$\tau_{xy} = 80 \quad \tau_{yz} = \tau_{zx} = 0$ MPa

Also determine the maximum normal stress and the maximum shear stress.

6.3.2 Plane Stress Revisited

$$\sigma_1 \neq \sigma_2 \neq 0 \qquad \sigma_z = \tau_{yz} = \tau_{zx} = 0$$

If PNSs in the x-y plane are σ_1 and σ_2, then the 3rd PNS $\sigma_3=0$. Even for this state shear stresses are present on all of the principal shear planes.

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} = \frac{|\sigma_2|}{2} \quad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{|\sigma_1|}{2} \quad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

Thus; *There is no state of plane stress as stresses occur on planes associated with choices of coordinate axes not in the x-y plane.*

6.3.2 Plane Stress Revisited, cont'd

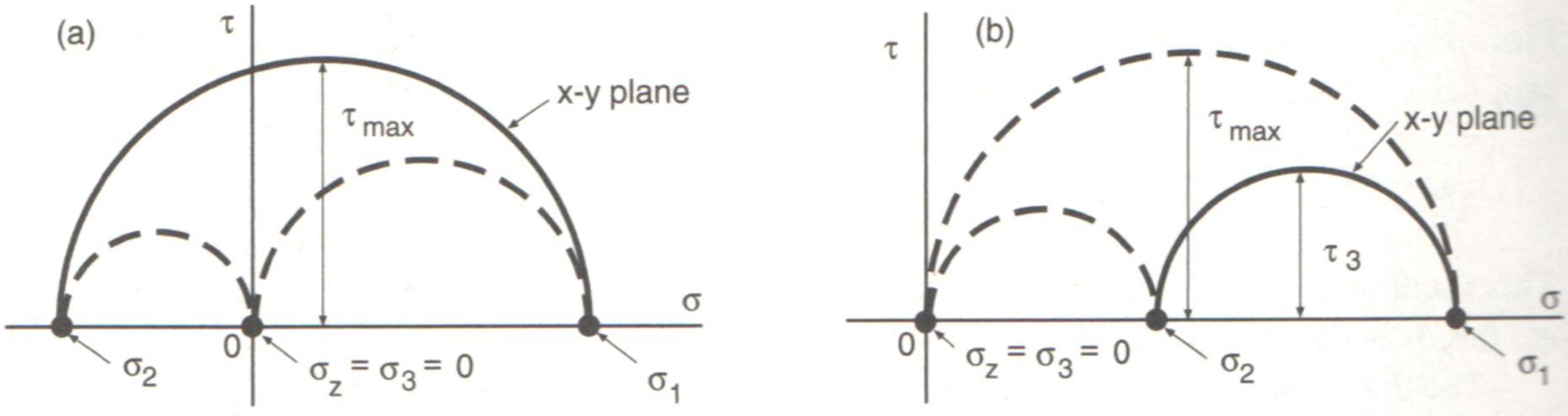


Figure 6.10 Plane stress in the *x-y* plane reconsidered as a three-dimensional state of stress. In case (a), the maximum shear stress lies in the *x-y* plane, but in case (b) it does not.

For (a), the PNSs in the x-y plane are of opposite sign, then the circle for the x-y plane is the largest, and τ_3 for the x-y plane is the maximum shear stress for all possible choices of coordinate axes. For (b), the PNSs for the x-y plane are of the same sign. In this case, one of the other circles is the largest. The radius of the largest circle is the maximum shear stress and this stress lies not in the x-y plane, but in the plane containing the direction of the largest PNS in the x-y plane and the z-axis.

Plane Stress Revisited, cont'd

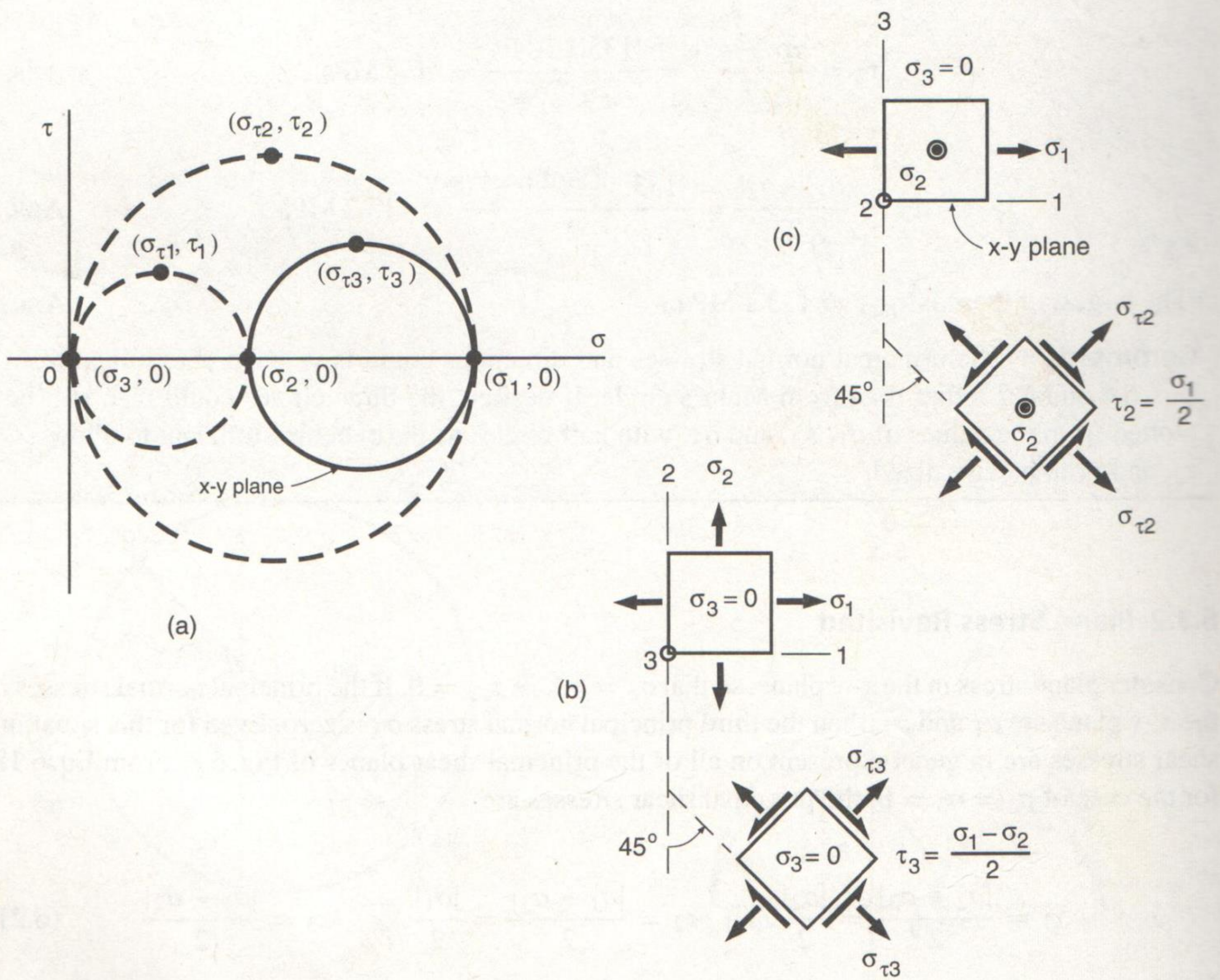


Figure 6.11 Plane stress with Mohr's circle not enclosing the origin, in which case the maximum shear stress is τ_2 oriented at 45° to the *x-y* plane.

Example 6.4

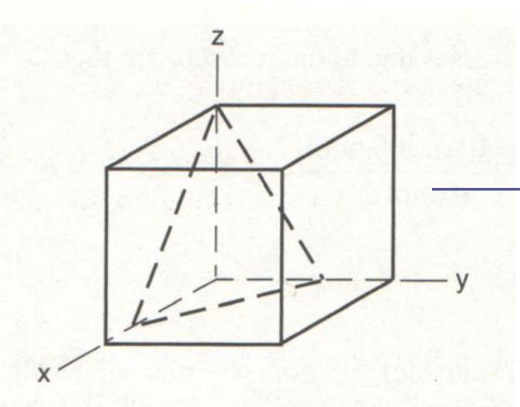
What is the maximum shear stress for the situation analyzed in Example 6.1 (σ_x=95MPa, σ_y=25MPa and τ_{xy}=20MPa).
Principal normal and shear stresses σ_1 = 100.3 MPa, σ_2 = 19.7 MPa, $\sigma_3 = \sigma_z$ = 0, and τ_3 = 40.3 MPa.

6.4 3-D State of Stress

In the general 3-D case, all six components of stress may be present.

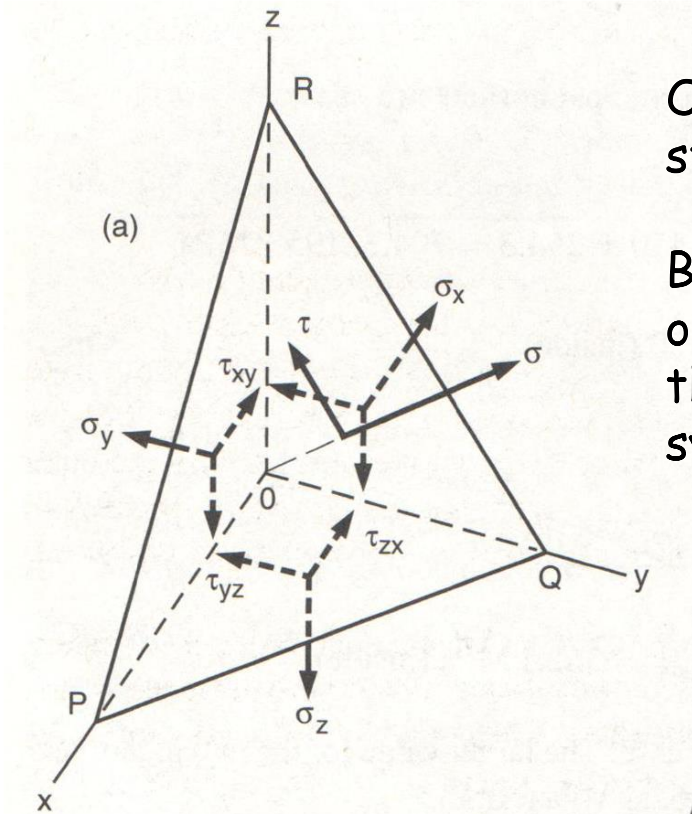
$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz} \; and \; \tau_{xz}$$

Transformation equations that permit to obtain any choice of coordinate axes in 3-D may be obtained.



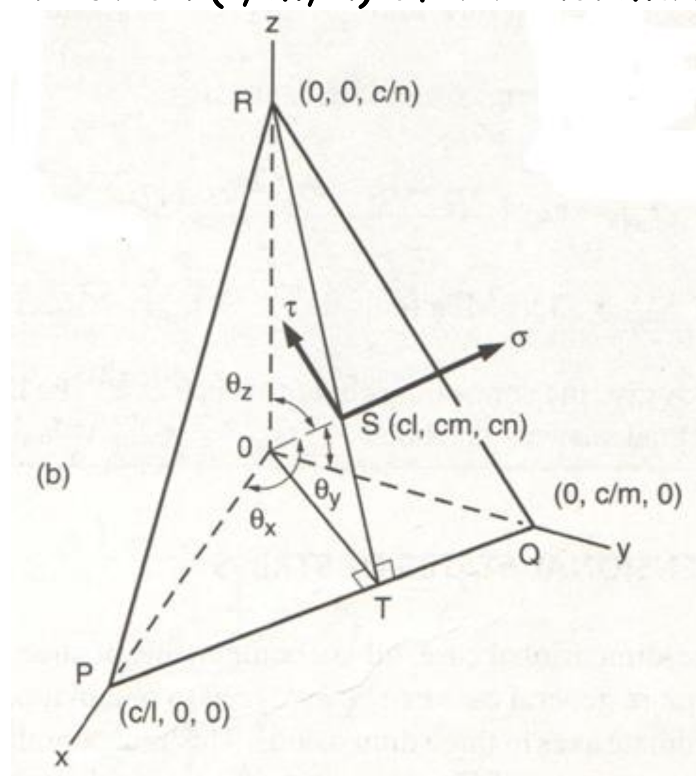
An oblique plane in a 3-D coordinate system

6.4 3-D State of Stress, cont'd



On the new oblique plane there is a normal (σ) stress and shear stress (τ)

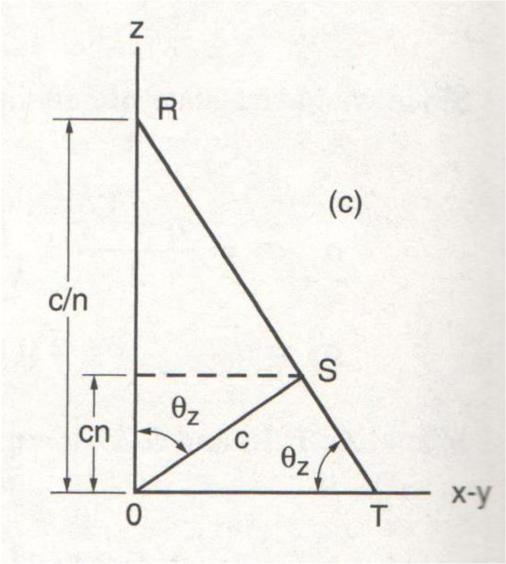
By applying equilibrium of forces to this portion of the cube, σ and τ can be evaluated in terms of the stresses on the original x-y-z coordinate system for any direction (l, m, n) of the normal.



Further analysis can be done to find PNSs and principal axes (1-2-3).

First, PNSs will be calculated by proceeding to the special case $\sigma=\sigma_i$ and $\tau=0$, where σ_i is any one of $\sigma_1, \sigma_2, \sigma_3$.

6.4.1 Geometry and axes

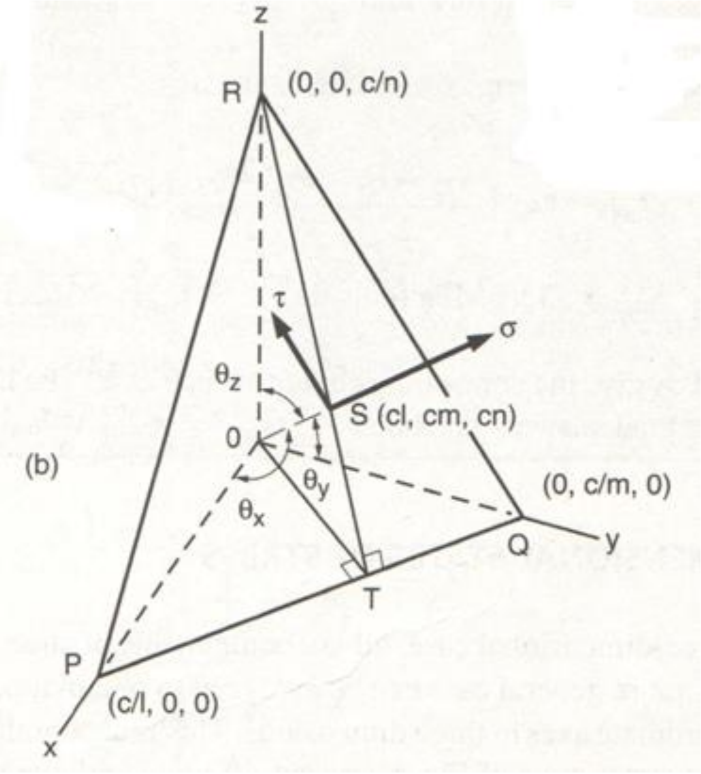


Direction cosines

$$l = Cos\theta_x$$

$$m = Cos\theta_y$$

$$n = Cos\theta_z$$



$$A_{yz} = lA_{PQR}$$

$$A_{zx} = mA_{PQR}$$

$$A_{xy} = nA_{PQR}$$

6.4.2 Principal normal stresses from equilibrium of forces

$$l_i\sigma_i A_{PQR} - \sigma_x A_{yz} - \tau_{xy} A_{zx} - \tau_{zx} A_{xy} = 0$$

$$-l_i\sigma_i A_{PQR} + \sigma_x l_i A_{PQR} + \tau_{xy} m_i A_{PQR} + \tau_{zx} n_i A_{PQR}$$

→ Above equation multiplied by (-1) and the areas of the 3 orthoghonal planes were invoked.

↓ Divide by A_{PQR} to obtain following eqs.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

Expanding this determinant gives a cubic equation

↓

$$\sigma^3 - \sigma^2(\sigma_x + \sigma_y + \sigma_z) + \sigma(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)$$
$$-(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0$$

6.4.2 Principal normal stresses from equilibrium of forces, cont'd

$$\sigma^3 - \sigma^2\left(\sigma_x + \sigma_y + \sigma_z\right) + \sigma\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2\right)$$
$$-\left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2\right) = 0$$

The cubic always has 3 real roots, which are the PNSs σ_1, σ_2, σ_3. The above eq. can be alternatively expressed as below. These are called stress invariants as they have the same values for all choices of coordinate system.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$\sigma_x + \sigma_y + \sigma_z = \sigma'_x + \sigma'_y + \sigma'_z = \text{constant}$$

→ I1 for xyz coordinate system is the same as the sum for the equivalent representation on any other coordinate system such as x'y'z'.

6.4.3 Directions for principal normal stresses

- To find directions first PNSs should be determined and then one of these (σ_1, σ_2, σ_3) substituted as σ_i into the below matrix, which are then simultaneously solved with $l_i^2+m_i^2+n_i^2=1$. To find all 3 principal axes, the process is repeated for each of σ_1, σ_2, σ_3.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

It is conventional to minimize negative signs.
Also three sets of direction cosines should represent a right-hand coordinate system. This can be accomplished by checking the vector cross product.
$(l_1, m_1, n_1) \times (l_2, m_2, n_2) = (l_3, m_3, n_3)$ - if this is not obeyed, then replace one direction cosine with its negative to satisfy the cross product.

Example 6.6:

At a point of interest in an engineering component, the stresses with respect to a convenient coordinate system are;

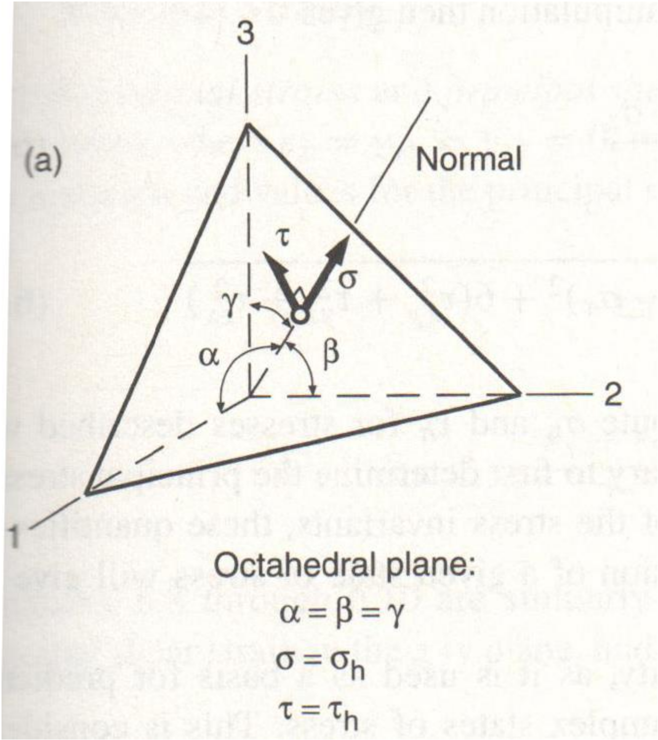
σ_x= 100MPa, σ_y= -60MPa, σ_z= 40MPa and τ_{xy}= 80MPa, τ_{yz}= τ_{zx}= 0

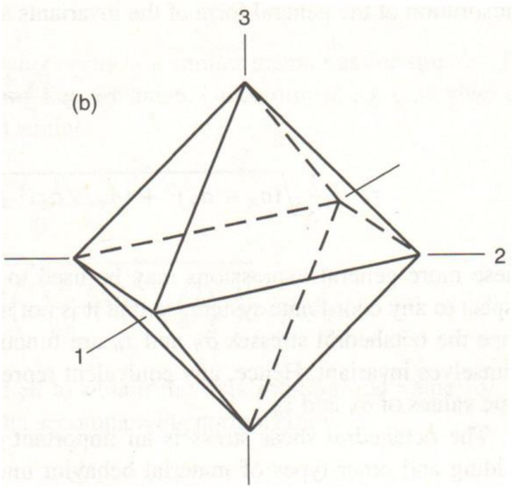
Determine the principal normal and shear stresses.

Example 6.7:

Find the direction cosines for each principal normal stress axis for the stress state of Ex. 6.6.

6.5 Stresses on the octahedral planes





Octahedral plane shown relative to the principal normal stress axes (a), and the octahedron formed by the similar such planes in all octants.

Special case:

$\alpha = \beta = \gamma$, the oblique plane intersects the principal axes at equal distances from the origin and is called the octahedral plane.

6.5 Stresses on the octahedral planes, cont'd

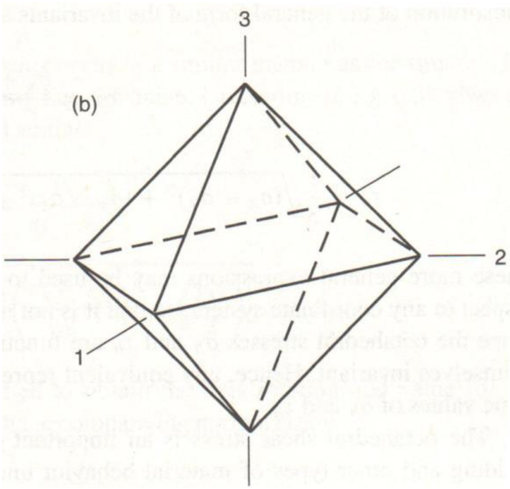
Based on equilibrium of forces, normal stress on this plane
σ_h = octahedral normal stress or hydrostatic stress ;

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Octahedral shear stress (τ_h) is evaluated as follows ;

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

6.5 Stresses on the octahedral planes, cont'd



The stresses on all 8 such planes are the same and are σ_h and τ_h. The opposing faces are correspond to a single plane.

σ_h and τ_h can also be written in terms of stress invariants.

$$\sigma_h = \frac{I_1}{3} \qquad \tau_h = \frac{1}{3}\sqrt{2(I_1^2 - 3I_2)}$$

Substituting of the general form of invariants and manipulation then gives following:

$$\sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \qquad \tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

6.6 Complex States of Strain

Equations developed for stress can be used for strain by changing the variables

$$\sigma_x \ \sigma_y \ \sigma_z \rightarrow \varepsilon_x \ \varepsilon_y \ \varepsilon_z$$

$$\tau_{xy} \ \tau_{yz} \ \tau_{xz} \rightarrow \frac{\gamma_{xy}}{2} \ \frac{\gamma_{yz}}{2} \ \frac{\gamma_{zx}}{2}$$

6.6.1 Principal strains:

$$For\ plane\ strain\ where\ \ \varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad \text{is obtained by modifying} \quad \tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad \text{is obtained by modifying}$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$Similarly:\ \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad and \quad \tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

are modified to obtain;

$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}} \quad and \quad \gamma_3 = \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \gamma_{xy}^2}$$

$$\varepsilon_{\gamma_3} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

As for the stress equations;

- ✓ θ is positive counterclockwise. Positive normal strains correspond to extension and negative ones to contraction.
- ✓ Positive shear strain causes a distortion corresponding to a positive shear stress in that the long diagonal of the resulting parallelogram has a positive slope.

These equations can be replaced by Mohr`s circle.

$$\sigma\, axis \rightarrow \varepsilon\, axis$$

$$\tau\, axis \rightarrow \gamma/2\, axis$$

For 3-D states of strain the principal strains can be obtained as shown below;

$$\begin{vmatrix} (\varepsilon_x - \varepsilon) & \gamma_{xy}/2 & \gamma_{zx}/2 \\ \gamma_{xy}/2 & (\varepsilon_y - \varepsilon) & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{yz}/2 & (\varepsilon_z - \varepsilon) \end{vmatrix} = 0$$

$$\gamma_1 = |\varepsilon_2 - \varepsilon_3|$$

$$\gamma_2 = |\varepsilon_1 - \varepsilon_3|$$

$$\gamma_3 = |\varepsilon_1 - \varepsilon_2|$$

6.6.2 Special Considerations for Plane Stress

- ✓ For cases of plane stress $\sigma_z=\tau_{yz}=\tau_{zx}=0$, the Poisson effect results in normal strains ε_z occurring in the out-of plane direction, so that the state of strain is 3 dimensional.
- ✓ If the material is isotropic, or if the material is orthotropic and a material symmetry plane is parallel to the x-y plane, no shear strains γ_{yz} or γ_{zx} occur.
- ✓ This creates a situation similar to that for stress where σ_z is present but $\tau_{yz}=\tau_{zx}=0$. Hence, one of the principal normal strains is $\varepsilon_z=\varepsilon_3$ and the other two can be obtained by using the below equation given before.

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

For isotropic, linear-elastic materials ε_z can be obtained from Hooke`s law in the form of;

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]\\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right]\end{aligned}\right\} \quad \text{Taking } \sigma_z = 0 \text{ and adding the 2 eq.s leads to;} \quad \sigma_x + \sigma_y = \frac{E}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Substitute this eq into $\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$ to obtain; $\varepsilon_z = \frac{-\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$

$$Since\ \tau_{yz} = \tau_{zx} = 0$$

$$From\ the\ previous\ equations\ of\ \gamma_{xy} = \frac{\tau_{xy}}{G}\ \gamma_{yz} = \frac{\tau_{yz}}{G}\ \gamma_{xz} = \frac{\tau_{xz}}{G}$$

$\gamma_{zx} = \gamma_{yz} = 0$, *and it is confirmed that* ε_z *is one of the principal normal strains*

Consider on orthotropic material under x-y plane stress where x-y plane of symmetry of the material. (This is the situation for most sheets and plates of composite materials). The strain ε_z is still one of the principal normal strains, as $\gamma_{xy} = \gamma_{xz} = 0$ holds in this case also. Hence, the strains in the x-y plane can still be used. However, ε_z <u>cannot be</u> obtained via following;

$$\varepsilon_z = \frac{-\upsilon}{1-\upsilon}(\varepsilon_x + \varepsilon_y)$$

A more general form of Hooke's law is needed!

$$\begin{bmatrix} 1/E_x & -\nu_{yx}/E_y & -\nu_{zx}/E_z & 0 & 0 & 0 \\ -\nu_{xy}/E_x & 1/E_y & -\nu_{zy}/E_z & 0 & 0 & 0 \\ -\nu_{xz}/E_x & -\nu_{yz}/E_y & 1/E_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{xy} \end{bmatrix}$$

Example:

At a point on a free (unloaded) surface of an engineering component made of an aluminum alloy, the following strains exist:

$$\varepsilon_x = -0.0005, \ \ \varepsilon_y = 0.0035 \ \ and \ \gamma_{xy} = 0.003$$

Determine the principal normal strains and shear strains. Assume that no yielding of the material has occurred.

Chapter 7. Yielding and Fracture under Combined Stresses

- ✓ Predicting the safe limits for use of a material under combined stresses requires the application of a failure criteria.
- ✓ Different failure criteria are available, some of which predict failure by yielding (yield criteria), others by fracture (fracture criteria).
- ✓ An effective value of stress that characterizes the combined stresses is calculated AND COMPARED with the yield or fracture strength of the material.
- ✓ An alternative to failure criteria based on stress is specifically analyze cracks in the material by the use of the special methods of fracture mechanics.

  In this chapter; materials are assumed to be isotropic and homogenous since failure criteria for anisotropic materials is a rather complex topic that is considered only to a limited extent.

7.2. General Form of Failure Criteria

In applying a yield criterion, the resistance of a material is given by its yield strength.

Failure criteria for isotropic materials: $$f\left(\sigma_1,\sigma_2,\sigma_3\right) = \underbrace{\sigma_c}_{\substack{\text{Failure strength} \\ \text{of the material}}} \qquad \left(at\ failure\right)$$

Failure is predicted to occur when a specific mathematical function f of the principal normal stresses is equal to the failure strength of the material, σ_c, as from a uniaxial test. Failure strength is either the yield strength σ_o, or the ultimate strength, σ_{ut} or σ_{uc} depending on whether yielding or fracture is of interest.

Requirements for a valid fracture criterion;

- ✓ It must give the same results regardless of the original choice of the coordinate system in a problem. This requirement is met if the criterion can be expressed in terms of the principal stresses. It is also met by any criterion where f is a mathematical function of one or more of the stress invariants (I_1, I_2, I_3).

- ✓ Consider a point in an engineering component where the applied loads result in particular values of the principal normal stresses, σ_1, σ_2 and σ_3 and where the materials property σ_c is known and also where a specific function f has been chosen. It is then useful to define an effective stress, which characterizes the state of applied stress.

$$\bar{\sigma} = f\left(\sigma_1, \sigma_2, \sigma_3\right) \quad then$$

$$Failure\,occurs\,when\,\bar{\sigma}\,is\,equal\,to\,\sigma_c \qquad \bar{\sigma} = \sigma_c\left(at\ failure\right)$$

$$Failure\,is\,not\,\exp ected\,if\ \bar{\sigma}\,is\,less\,than\,\sigma_c \qquad \bar{\sigma} < \sigma_c\ \left(no\ failure\right)$$

$$the\,safety\ factor\,against\ failure; \qquad X = \frac{\sigma_c}{\bar{\sigma}}$$

7.3. Maximum Normal Stress Fracture Criterion

- ✓ FAILURE is EXPECTED when the LARGEST PRINCIPAL NORMAL STRESS REACHES the UNIAXIAL STRENGTH of the MATERIAL. (Reasonably successful in predicting fracture of brittle materials under tension-dominated loading.)

$$\sigma_u = Max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) \qquad (at\ fracture)$$

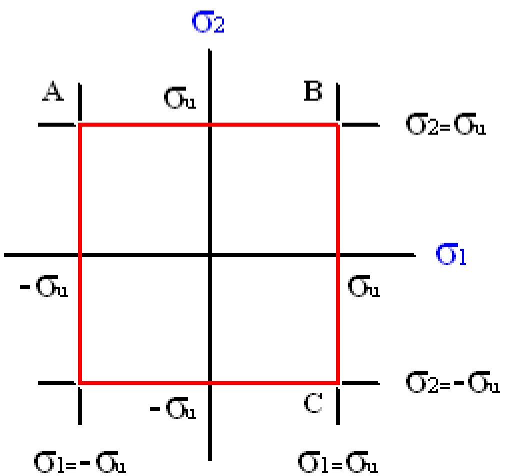
$$\bar{\sigma}_N = Max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right)$$

- ✓ Fracture is expected when effective normal stress is σ_u but not when it is less and the safety factor against fracture

$$X = \sigma_u / \bar{\sigma}_N$$

7.3.1. Graphical representation of the Maximum Normal Stress Fracture Criterion

For plane stress $\sigma_3 = 0$



- ✓ Any combination of σ_1 and σ_2 that plots within the square box is safe and any on its perimeter corresponds to fracture.
- ✓ The box is the region that satisfies $Max(|\sigma_1|, |\sigma_2|) \leq \sigma_u$

Equations for the 4 straight lines;

$$\sigma_1 = \sigma_u, \quad \sigma_1 = -\sigma_u, \quad \sigma_2 = \sigma_u, \quad \sigma_2 = -\sigma_u$$

7.3.1. Graphical representation of the Maximum Normal Stress Fracture Criterion, cont'd

- ✓ For the general case, where all the principal normal stresses may have nonzero values, safe region is bounded by;

$$\sigma_1 = \pm\sigma_u \qquad \sigma_2 = \pm\sigma_u \qquad \sigma_3 = \pm\sigma_u$$

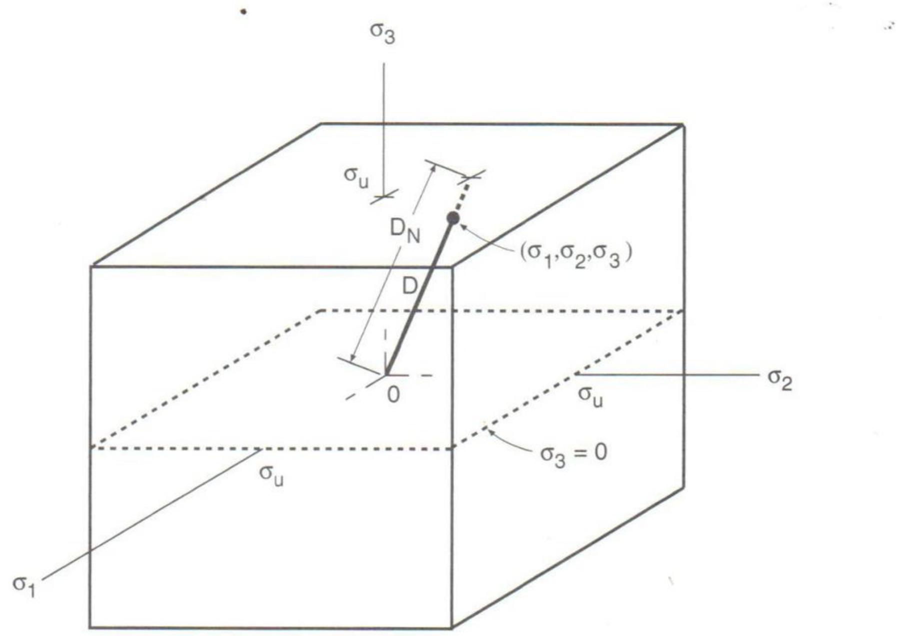
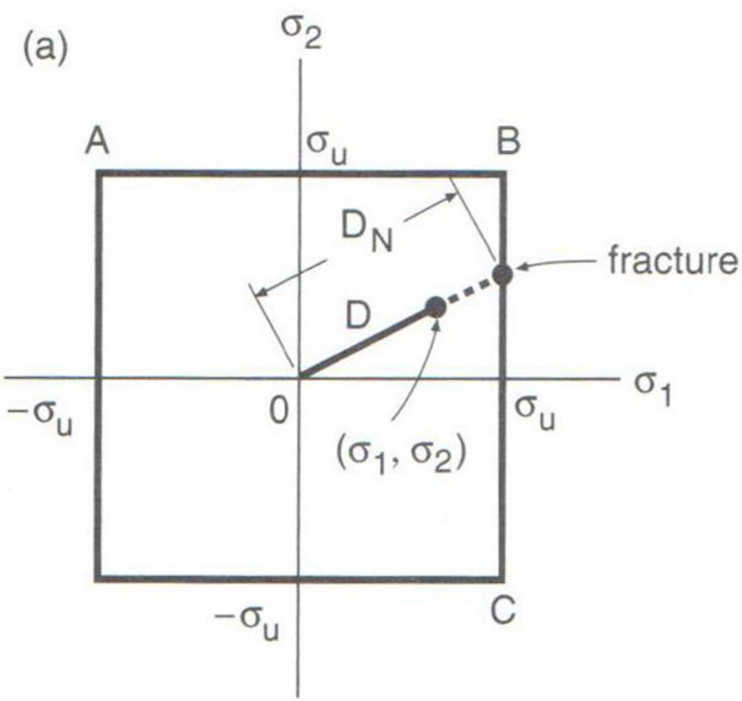


Figure 7.3 Three-dimensional failure surface for the maximum normal stress fracture criterion.

- ✓ The failure surface is simply a cube.

7.3.1. Graphical representation of the Maximum Normal Stress Fracture Criterion, cont'd

✓ For 2-D case following figure should be considered



Safety factor could be schematically represented.

D represent the applied surface
D_N since extends to fracture surface represent the limit condition.
Then, safety factor against fracture can be obtained as $X = D_N/D$

✓ See next slide for further explanation!

7.3.1. Graphical representation of the Maximum Normal Stress Fracture Criterion, cont'd

- ✓ For brittle materials ultimate strength in compression is usually considerably higher than ultimate strength in tension. Hence for engineering use the fracture criterion needs to be restricted to tension dominated loading, which can be accomplished with the following formulas.

$$\bar{\sigma}_{NT} = Max(\sigma_1, \sigma_2, \sigma_3), \quad \mathrm{X} = \frac{\sigma_{ut}}{\bar{\sigma}_{NT}}$$

$$where \quad \bar{\sigma}_{NT} \rangle 0, \ \text{and} \ |\sigma_{\max}| \rangle |\sigma_{\min}|$$

This means that maximum principal normal stress be both positive and larger in magnitude than any principal normal stress that is negative (compressive). Therefore, D_N strikes the failure criterion withing ABC for 2-D case and strikes one of the positive facing sides of the cube for the 3-D case.

Example 7.1:

A sample of gray cast iron is subjected to the state of generalized plane stress of Ex.6.3. Gray cast iron normally behaves in a brittle manner and this particular material has ultimate strengths in tension and compression of σ_{ut}=214 and $|\sigma_{uc}|$=770 MPa, respectively. What is the safety factor against fracture?

$$\sigma_x = 100 \qquad \sigma_y = 100 \qquad \sigma_z = 40MPa$$

$$\tau_{xy} = 80 \qquad \tau_{yz} = \tau_{zx} = 0MPa$$

Principal normal stresses are calculated;

$$\sigma_1 = 133.1 \qquad \sigma_y = -93.1 \qquad \sigma_z = 40MPa$$

$$\bar{\sigma} = Max(\sigma_1, \sigma_2, \sigma_3) = Max\left(\underset{+Tension}{133.1}, \underset{-Compression}{-93.1}, 40\right) = 133.1MPa$$

$$X = \frac{\sigma_{ut}}{\bar{\sigma}} = \frac{214}{133.1} = 1.61$$

tension dominated case

$$\underset{133.1}{|\sigma_{max}|} > \underset{93.1}{|\sigma_{min}|}$$

7.4. Maximum Shear Stress Yield Criterion (Tresca Criterion)

- ✓ Yielding of ductile materials is often predicted to occur when the maximum shear stress on any plane reaches a critical value τ_0, which is a property;

$$\tau_0 = \tau_{max} \text{ (at yielding)}$$

- ✓ For metals, such an approach is logical because the mechanism of yielding on a microscopic size scale is the slip of crystal planes, which is a shear deformation that is expected to be controlled by a shear stress.

- ✓ Recall that max. shear stress is the largest of the 3 principal shear stresses which act on planes oriented at 45° relative to the principal normal stress axes.

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} \qquad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} \qquad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

7.4. Maximum Shear Stress Yield Criterion (Tresca Criterion), cont'd

Hence this yield criterion can be stated as follows;

$$\tau_o = Max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right) \quad (at\ yielding)$$

If τ_0 is represented by means of σ_0 $\sigma_1 = \sigma_0$ $\sigma_2 = \sigma_3 = 0$

Substitute these values into yield criterion to obtain $\tau_0 = \sigma_0/2$

Then modify above equation to obtain following;

$$\frac{\sigma_o}{2} = Max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right) \quad (at\ yielding)$$

or

$$\sigma_o = Max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_1 - \sigma_3|\right) \quad (at\ yielding)$$

7.4. Maximum Shear Stress Yield Criterion (Tresca Criterion), cont'd

The effective stress; $\bar{\sigma}_s = Max\left(\left|\sigma_1 - \sigma_2\right|, \left|\sigma_2 - \sigma_3\right|, \left|\sigma_1 - \sigma_3\right|\right)$

The safety factor against yielding is then $X = \dfrac{\sigma_o}{\bar{\sigma}_s}$

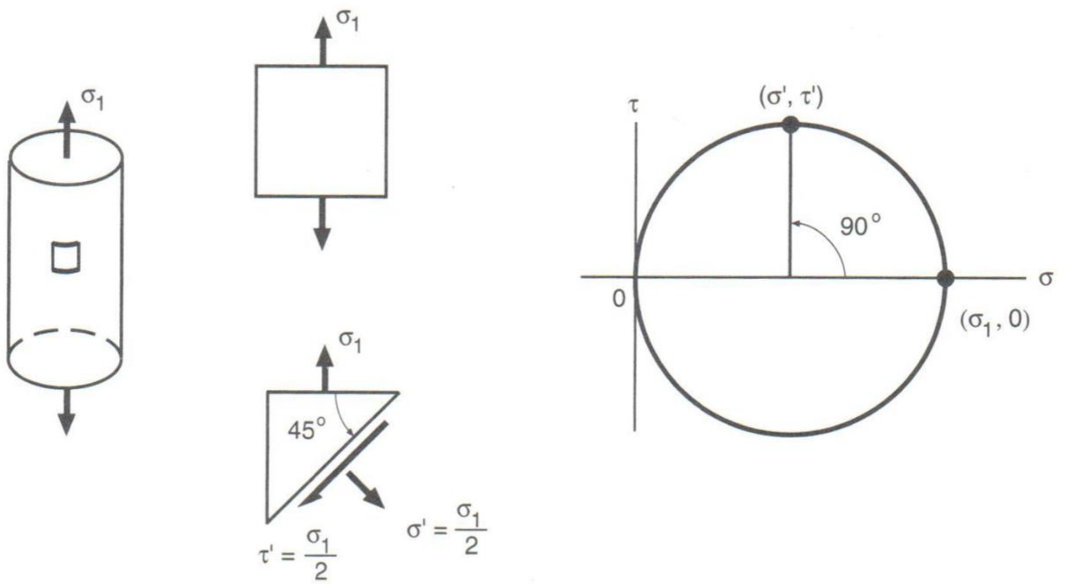
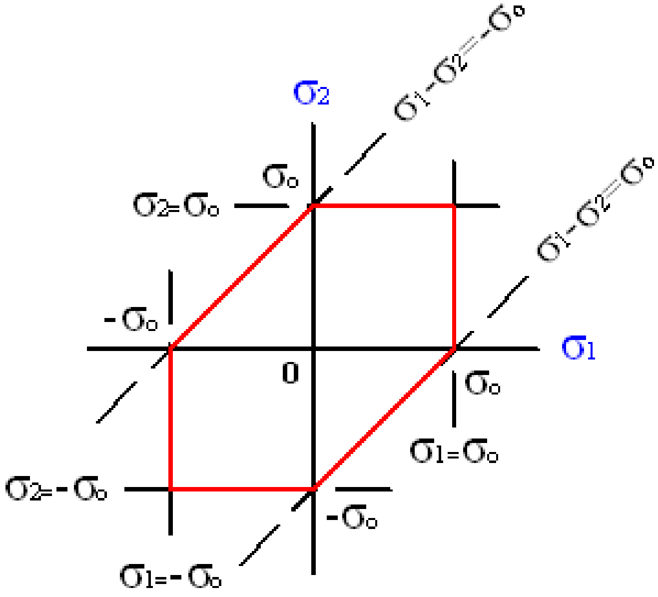


Figure 7.4 The plane of maximum shear in a uniaxial tension test.

7.4.2. Graphical representation of the Maximum Shear Stress Yield Criterion (Tresca Criterion)

For plane stress $\sigma_3 = 0$



Points on the hexagon correspond to yielding and points inside are safe

$$\sigma_o = Max\left(|\sigma_1 - \sigma_2|, |\sigma_2|, |\sigma_1|\right)$$

The region of yielding bounded by the lines

$$\sigma_1 - \sigma_2 = \pm\sigma_o \qquad \sigma_2 - \sigma_3 = \pm\sigma_o \qquad \sigma_1 - \sigma_3 = \pm\sigma_o$$

7.4.2. Three dimensional failure surface for the maximum shear stress yield criterion

$$\sigma_1 - \sigma_2 = \pm\sigma_o \qquad \sigma_2 - \sigma_3 = \pm\sigma_o \qquad \sigma_1 - \sigma_3 = \pm\sigma_o$$

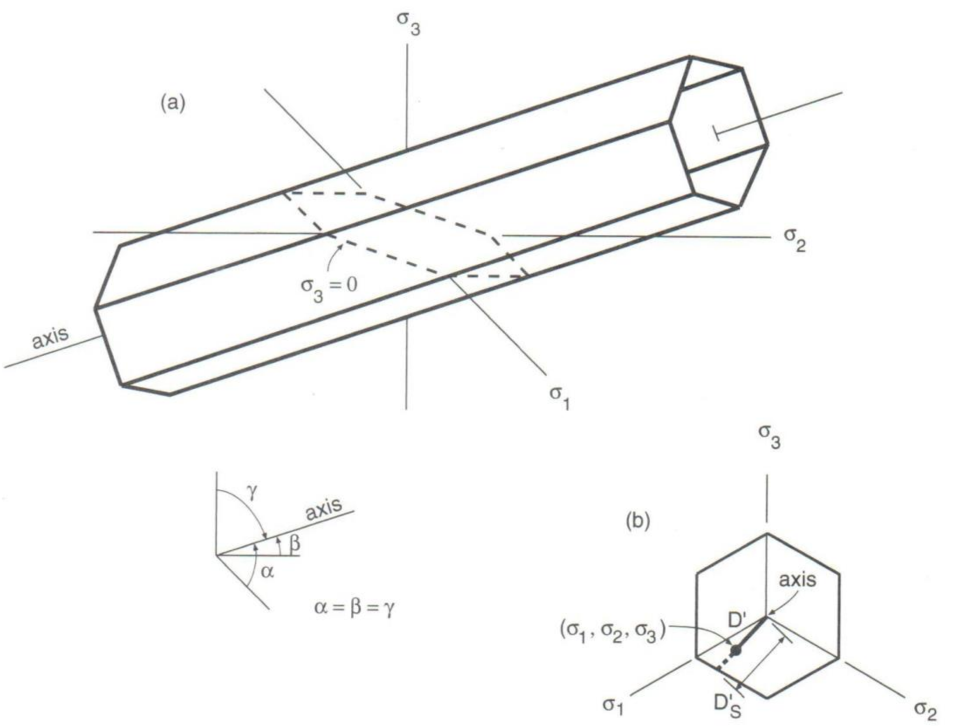


Figure 7.6 Three-dimensional failure surface for the maximum shear stress yield criterion.

These 3 pairs of plane form a tube with a hexagonal cross-section. The axis of the tube is the line $\sigma_1=\sigma_2=\sigma_3$

7.4.3. Hydrostatic stresses and the maximum shear stress criterion

Consider a special case of stress $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h$

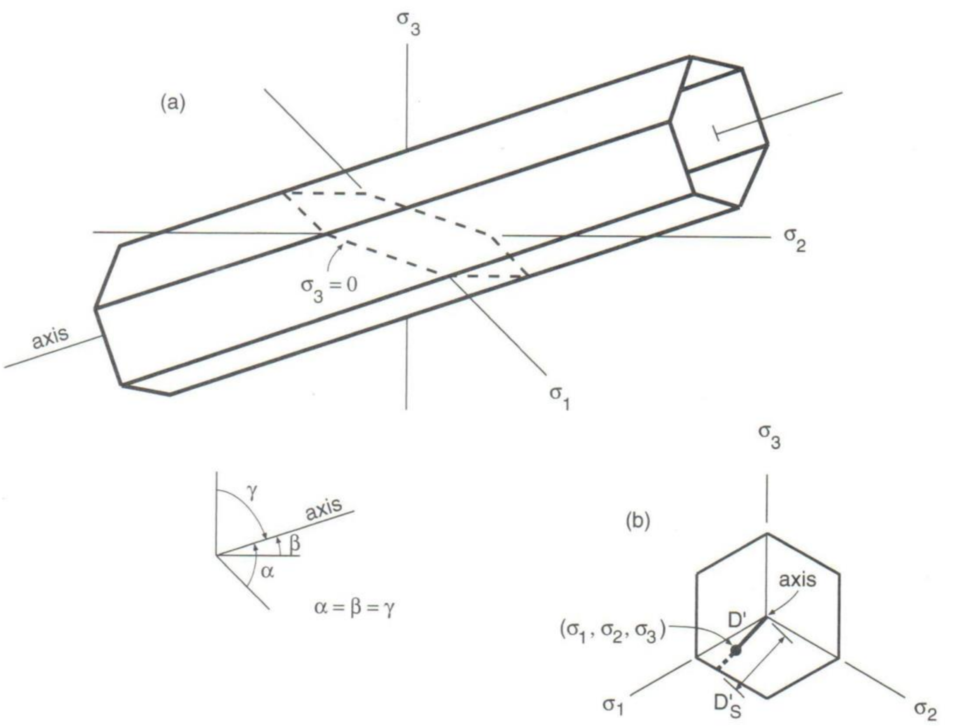


Figure 7.6 Three-dimensional failure surface for the maximum shear stress yield criterion.

Example: a simple pressure loading p, so that $\sigma_h = -p$

This corresponds to a point along the axis of hexagonal cylinder. For a such point the effective stress $\bar{\sigma}_s$ is always 0 and the safety factor against yielding is thus infinite.

7.4.3. Hydrostatic stresses and the maximum shear stress criterion, cont'd

Hence the maximum shear stress criterion predicts that hydrostatic stress alone does not cause yielding. This result is surprising but in agreement with experimental findings. It is likely that brittle fracture without yielding would occur at a high stress level even in normally ductile materials.

7.5. Octahedral Shear Stress Yield Criterion (or von Mises or distortion energy criterion)

Yielding occurs when the shear stress on the octahedral planes reaches the critical value. $\tau_h = \tau_{h0}$

$$\tau_h = \tau_{h0} \quad \text{(at yielding)}$$

where,

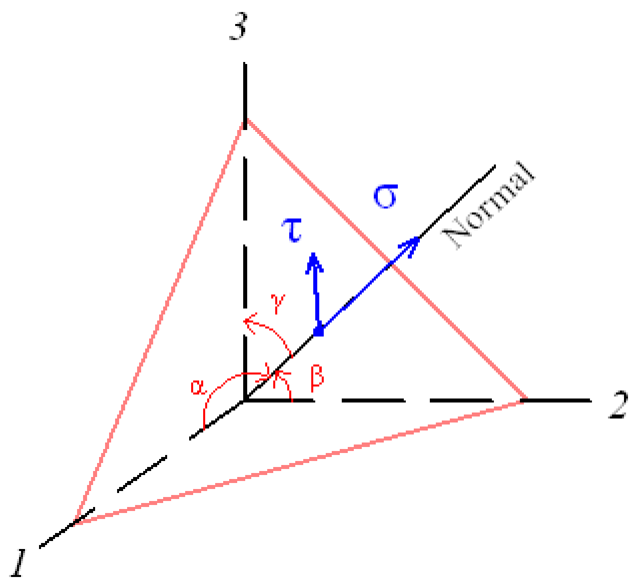
τ_{h0} is the value of octahedral stress τ_h necessary to cause yielding.

- ✓ An alternative to maximum shear criterion
- ✓ Often used for ductile materials

7.5. Octahedral Shear Stress Yield Criterion (or von Mises or distortion energy criterion), cont'd

- ✓ Since hydrostatic stress σ_h is observed not to affect yielding, it is logical to find the plane where this occurs as the normal stress and then to use the remaining stress τ_h as the failure criterion.
- ✓ Another justification is to note that, although yielding is caused by shear stresses, τ_{max} occurs on only 2 planes in the material, whereas τ_h is never very much smaller and occurs on 4 planes.
- ✓ Hence on a statistical basis, τ_h has a greater chance of finding crystal planes that are favorably oriented for slip and this may overcome its disadvantage of being slightly smaller than τ_{max}.

7.5. Octahedral Shear Stress Yield Criterion (or von Mises or distortion energy criterion), cont'd



$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad \left(\begin{array}{c} Octahedral\ normal\ stress \\ or\ the\ hydrostatic\ stress \end{array} \right)$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_{ho} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Octahedral plane

$$\alpha = \beta = \gamma$$

$$\sigma = \sigma_h$$

$$\tau = \tau_h$$

- ✓ As was done for the maximum shear stress criterion, it is useful to express the critical value in terms of the yield strength from a tension test.
- ✓ Substitution of the uniaxial stress state with $\sigma_1 = \sigma_0$ and $\sigma_2 = \sigma_3 = 0$ into the octahedral shear criterion gives;

$$\tau_{ho} = \frac{\sqrt{2}}{3}\sigma_o$$

7.5.2. Graphical representation of the Octahedral Shear Stress Yield Criterion, cont'd

The plane on which the uniaxial stress acts is related to the octahedral plane by a rotation through the angle α as seen on the figure below.

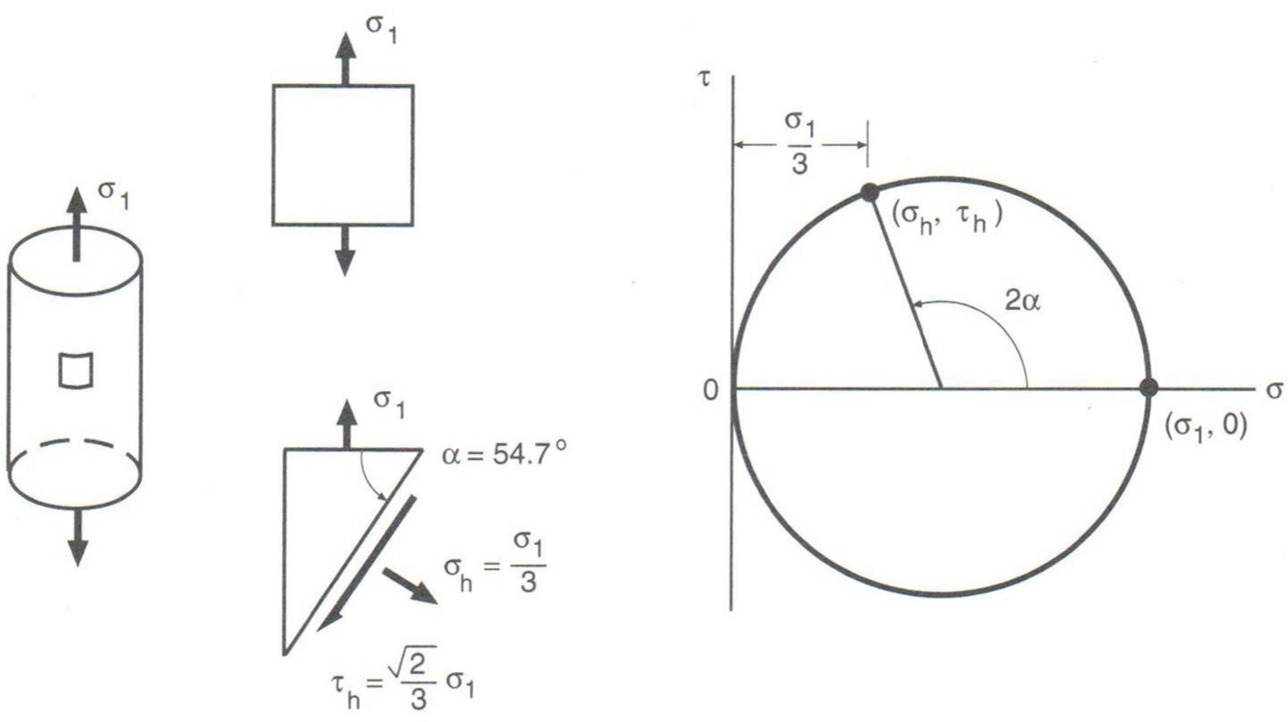


Figure 7.7 The plane of octahedral shear in a uniaxial tension test.

7.5.2. Graphical representation of the Octahedral Shear Stress Yield Criterion, cont'd

Other related formulae.

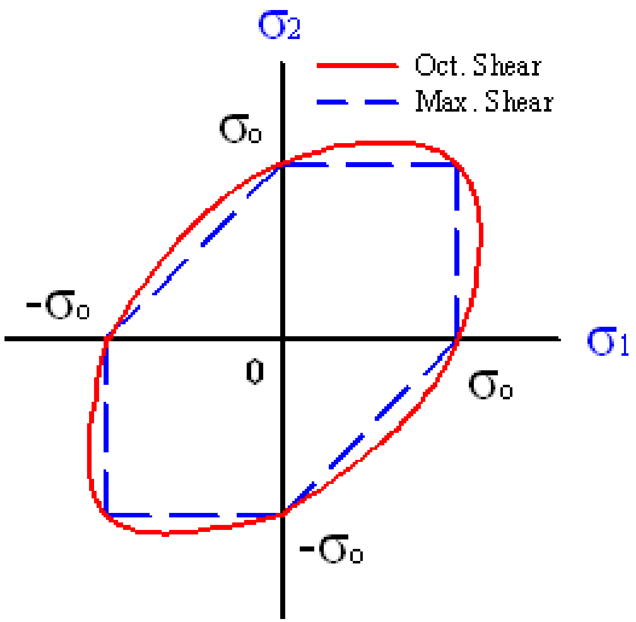
$$\sigma_0 = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

$$\bar{\sigma}_H = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

$$X = \sigma_0 / \bar{\sigma}_H$$

$$\bar{\sigma}_H = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)}$$

7.5.2. Graphical representation of the Octahedral Shear Stress Yield Criterion, cont'd



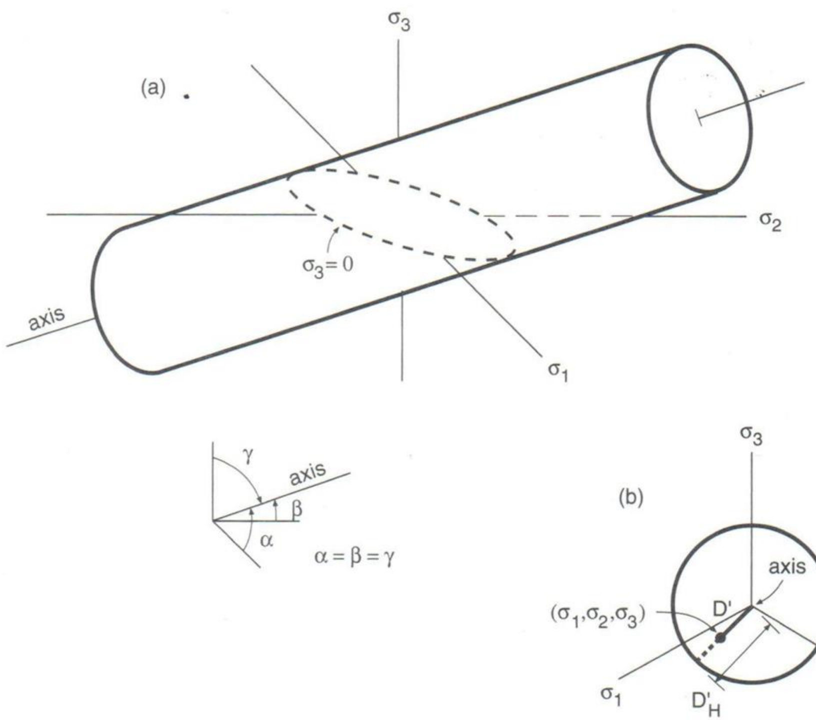


Figure 7.9 Three-dimensional failure surface for the octahedral shear stress yield criterion.

The 3 criteria discussed so far, may be considered to be the basic ones among a larger number that are available.

7.5.2. Graphical comparison of the Octahedral Shear Stress Yield and the Maximum Shear Stress criteria

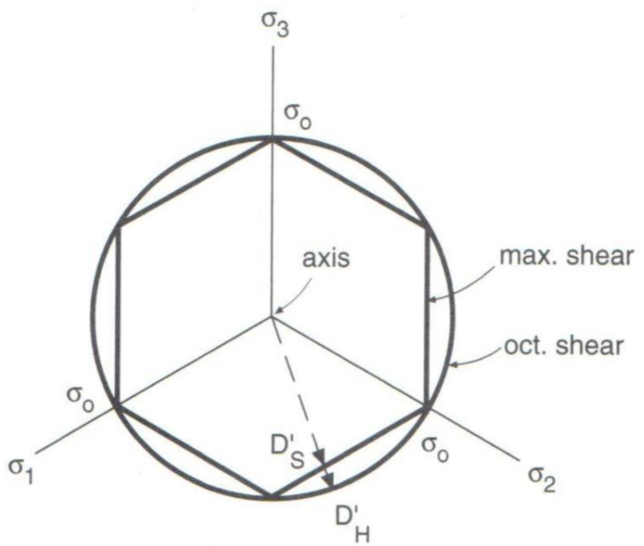


Figure 7.10 Comparison of yield surfaces for the maximum shear and octahedral shear stress criteria.

Example 7.2

Consider the simple pipe with closed ends with wall thickness 10mm and inner diameter 0.60m subjected to 20MPa internal pressure and a torque of 1200kNm. What is the safety factor against yielding at the inner wall if the pipe is made of the 18 Ni maraging steel of Table 4.2. (Yield stress for 18 Ni steel is 1791 MPa).

Example 7.3

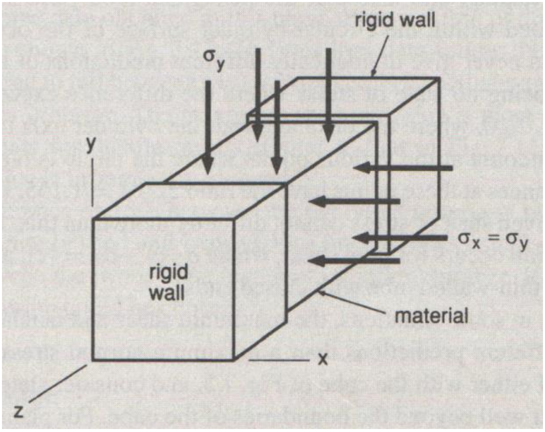
A solid shaft of diameter d is made of AISI 1020 steel (as rolled) and is subjected to an axial load of 200kN and a torque of 1.50kNm. a) what is the safety factor against yielding if the diameter is 50mm? b) for the situation of (a), what adjusted value of diameter is required to obtain safety factor agianst yielding of 2.0?

Example 7.4

Repeat example 7.2., except use the octahedral shear stress yield criterion.

Example 7.5

A block of material is subjected to equal and compressive stresses in the x and y directions, and it is confined by a rigid die so that it cannot deform in the z direction, as shown in the below figure. Assume that there is no friction against the die and also that the material behaves in an elastic, perfectly plastic manner, with uniaxial yield strength σ_0. a) determine the stress $\sigma_x = \sigma_y$ necessary to cause yielding, expressing this as a function of σ_0 and elastic constants of the material. b) what is the value of σ_y at yielding if the material is an aluminum alloy with uniaxial yield strength σ_0 = 300MPa and elastic constants as in Table 5.2 (υ=0.345).



7.6 Discussion of the basic failure criteria

7.6.1 Comparison of failure criteria

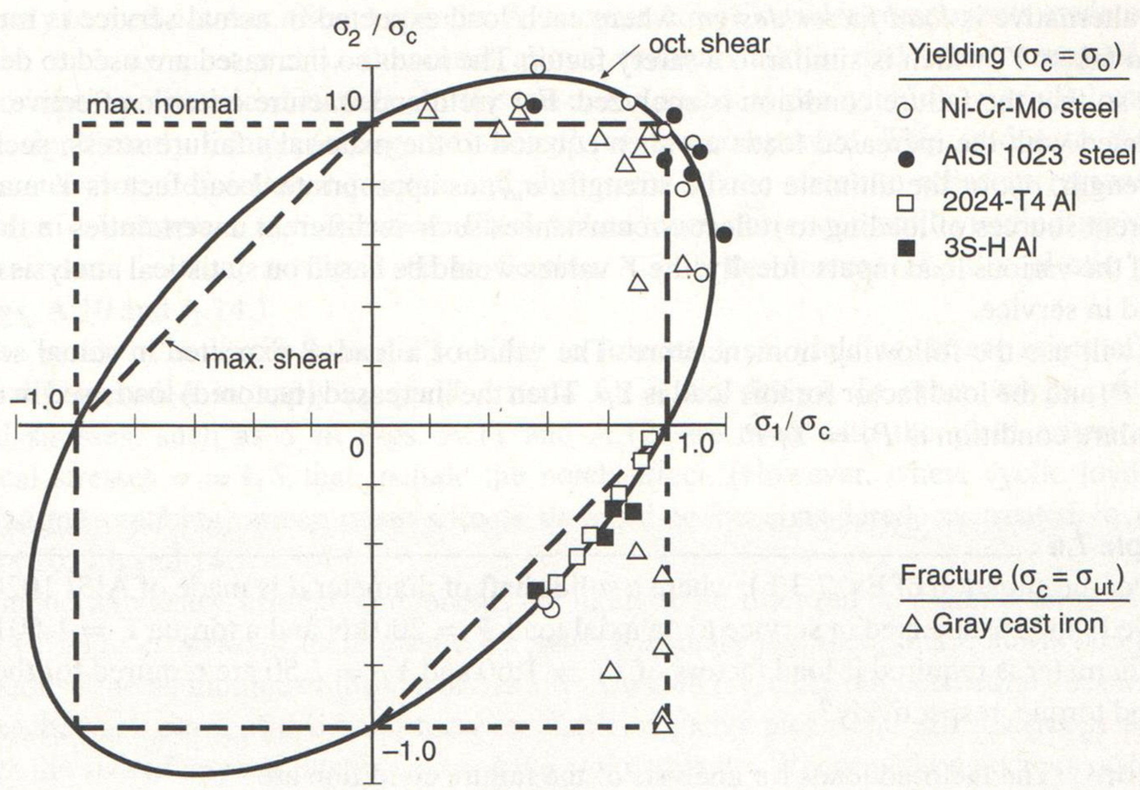


Figure 7.11 Plane stress failure loci for three criteria. These are compared with biaxial yield data for ductile steels and aluminum alloys, and also with biaxial fracture data for gray cast iron. (The steel data are from [Lessells 40] and [Davis 45], the aluminum data from [Naghdi 58] and [Marin 40], and the cast iron data from [Coffin 50] and [Grassi 49].)

Chapter 8. Fracture of Cracked Members

- ✓ The presence of a crack → weakens the material
- ✓ Fracture may occur at stresses below the material`s yield strength where failure would not normally be expected.

Before FRACTURE MECHANICS

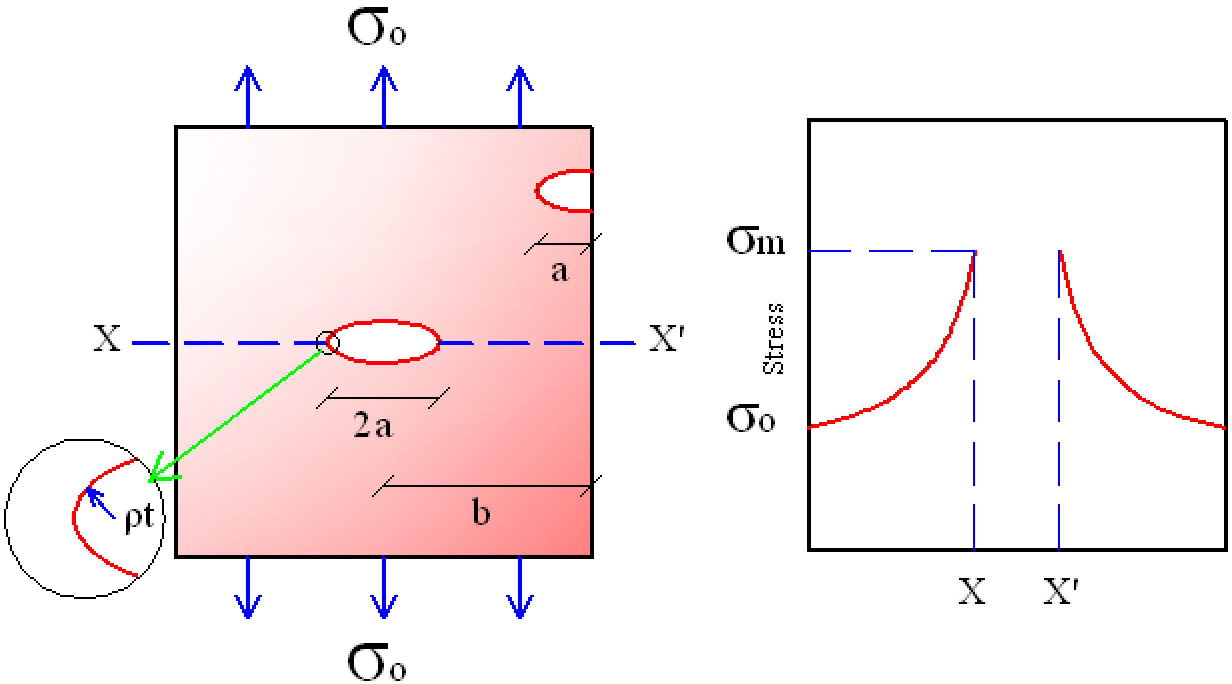
- ✓ Engineering design was based on → tension, compression and bending tests along with failure criteria for nominally uncracked material.

Calculated (theoretical) strength >> Measured strength

??

(Due to existence of cracks and flaws)

8.2.1 Cracks as Stress Raisers



$$\sigma_m = 2\sigma_o \left(\frac{a}{\rho_t} \right)^{1/2}$$

σ_0 : magnitude of the nominal stress

ρ_t:radius of curvature of the crack tip

a: length of a surface crack or half length of an internal crack.

K: Stress intensity factor

$$\underset{\text{A representation of amplification of stress which occurs due to crack}}{K} = \frac{\sigma_m}{\sigma_o} = 2\left(\frac{a}{\rho_t} \right)^{1/2} \quad \text{small } \rho \rightarrow \text{large} \left(\frac{a}{\rho_t} \right)^{1/2} \rightarrow \text{large stress amplification}$$

8.2.2 Behavior at crack tips in real materials

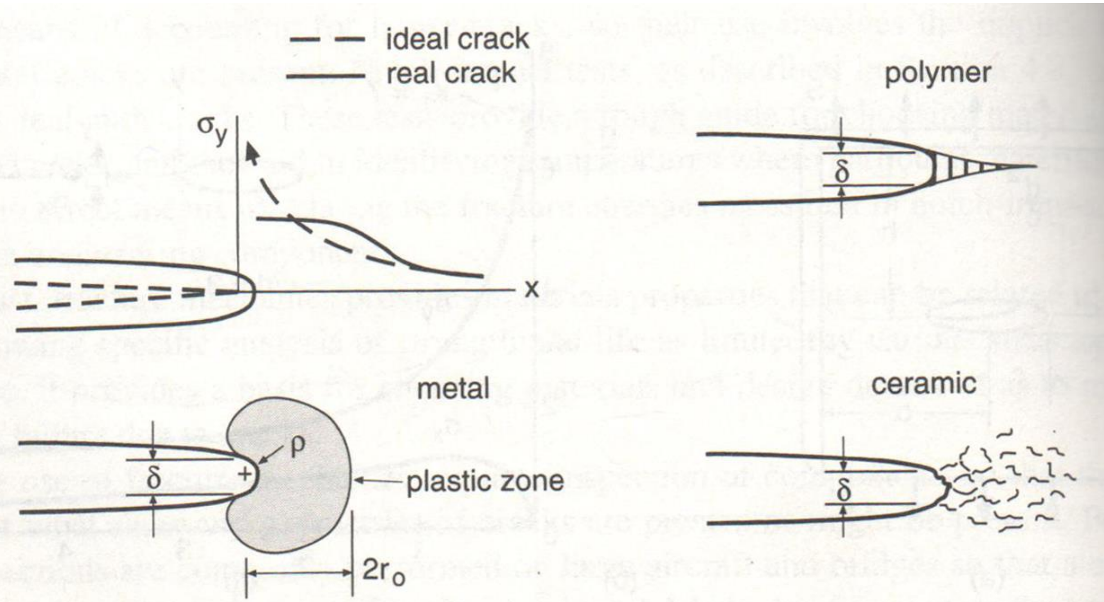


Figure 8.4 Finite stresses and nonzero radii at tips of cracks in real materials. A region of intense deformation forms due to plasticity, crazing, or microcracking.

8.2.3 Effects of cracks on strength

K : stress intensity factor – a measure of the severity of a crack situation as affected by crack size, stress and geometry.

A given material can resist a crack without brittle fracture occurring as long as this K is below a critical value K_c

K_c = fracture toughness (affected by temperature, loading rate and secondarily by the thickness of the member)

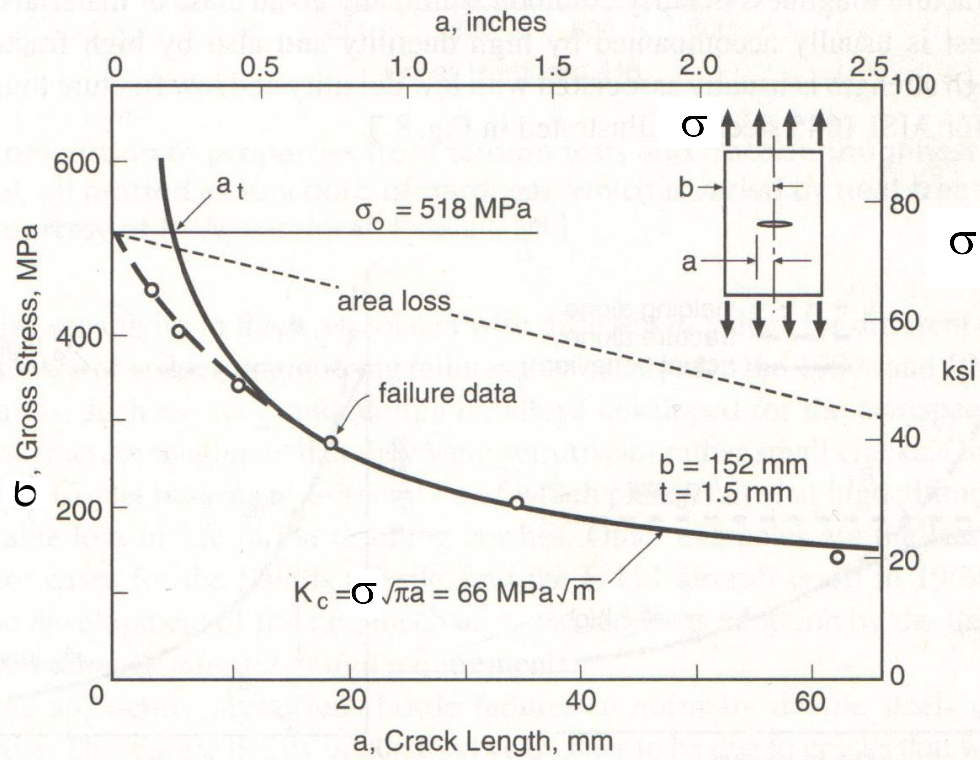
K_{Ic} called the plane strain fracture toughness.

K_{Ic} is a measure of a given material's resistance to a crack.

8.2.3 Effects of cracks on strength, cont'd

$K = \sigma\sqrt{\pi a}$ only when (a << b)

The critical value of the applied stress or the critical stress $\sigma_c = K_c / \sqrt{\pi a}$



Compare the theoretical and experimental curves

Figure 8.5 Failure data for cracked plates of 2014-T6 Al tested at −195°C. (Data from [Orange 67].)

8.2.4 Effects of Cracks on Brittle vs. Ductile Behavior

Crack length (a_t) required for failure (transition crack length).

$$a_t = \frac{1}{\pi}\left(\frac{K_c}{\sigma_o}\right)^2$$

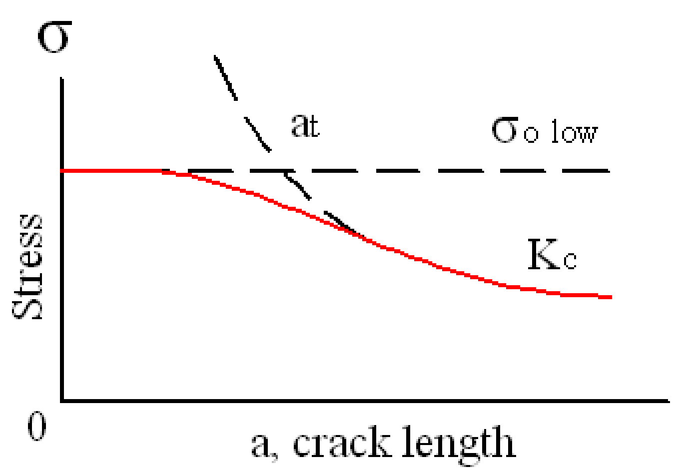
(Valid for wide, center-cracked plate)
a_t = Transition crack length
σ_o = Yield strength

If $a > a_t$ → material will fail by brittle fracture rather than by yielding.

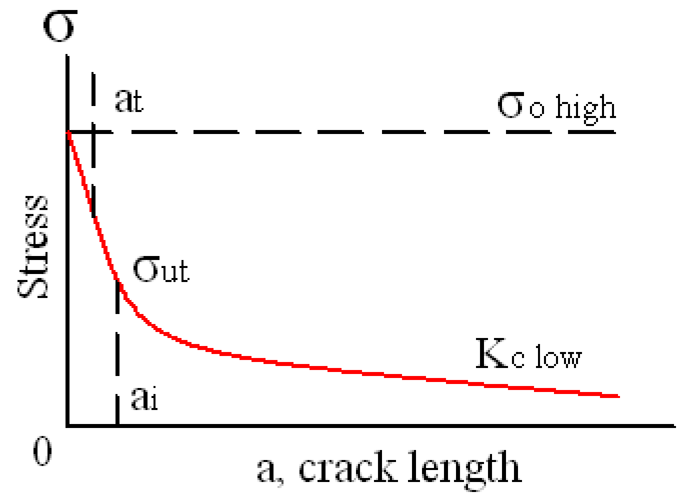
Thus, if cracks of length around or grater than the a_t of a given material are likely to be present, fracture mechanics should be employed in design. Conversly for crack lengths below a_t, yielding dominated behavior is expected, so that there will little or no strength reduction due to the crack.

8.2.4 Effects of Cracks on Brittle vs. Ductile Behavior

Low σ_o, High K_c material



High σ_o, Low K_c material



Moderate size cracks may not affect the low-strength material but they may severally limit the usefulness of the high-strength one.

Low strength in a tension test → Mostly accompanied by high ductility and high fracture toughness.

High strength in a tension test → Mostly accompanied by low ductility and low fracture toughness.

8.2.4 Effects of Cracks on Brittle vs. Ductile Behavior

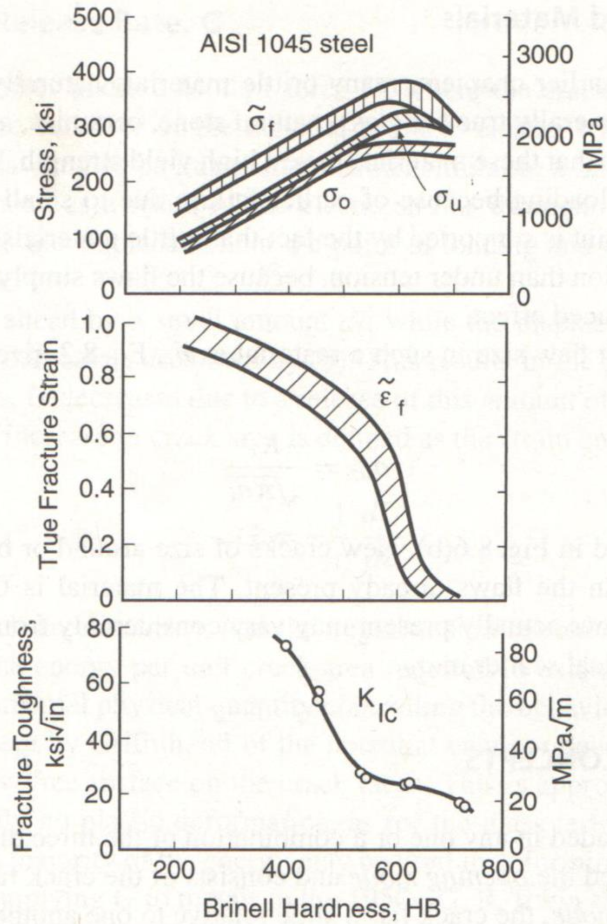


Figure 8.7 Comparison of properties from tension tests and fracture toughness tests for AISI 1045 steel, all plotted as functions of hardness, which is varied by heat treatment. (Illustration courtesy of R. W. Landgraf, Howell, MI.)

8.2.4 Effects of Cracks on Brittle vs. Ductile Behavior

Several failures occurred in the past due to the usage of materials with low fracture toughness.

Liberty ships of US

8.2.5 Internally flawed materials

Calculated (theoretical) strength >> Measured strength
(Due to existence of cracks and flaws)

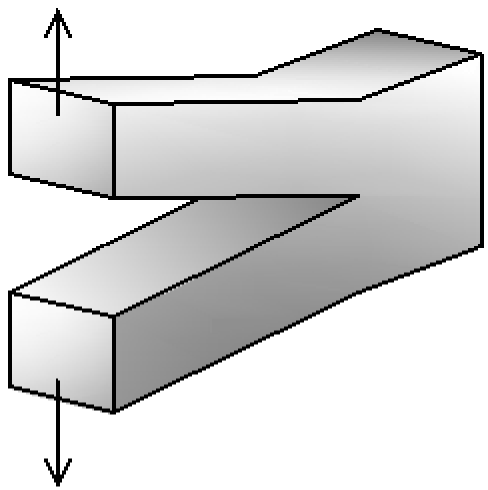
Remember compressive strength of brittle materials considerably higher than tensile strength, because the flaws simply close under compression

If inherent flaw size in a material is a_i, then the ultimate tensile strength can be calculated as follows:

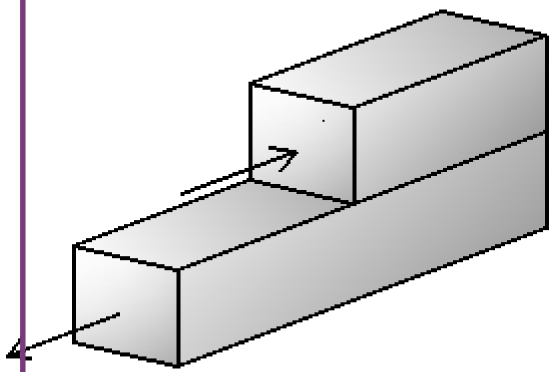
$$\sigma_{ut} = \frac{K_c}{\sqrt{\pi a_i}}$$

8.3 Mathematical concepts

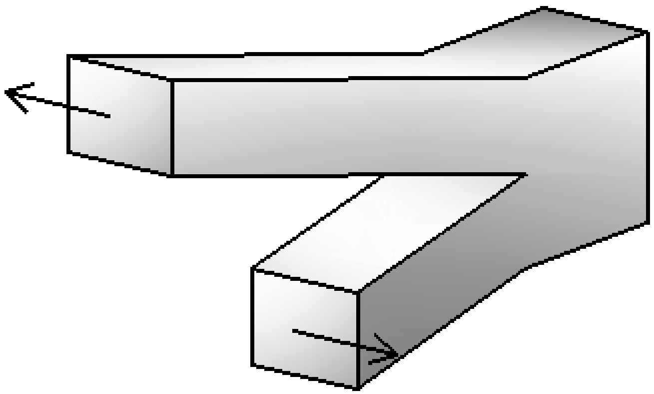
Modes of Failure



Mode I: Opening or tensile mode
Mode I is caused by tension loading. Crack faces simply move apart.

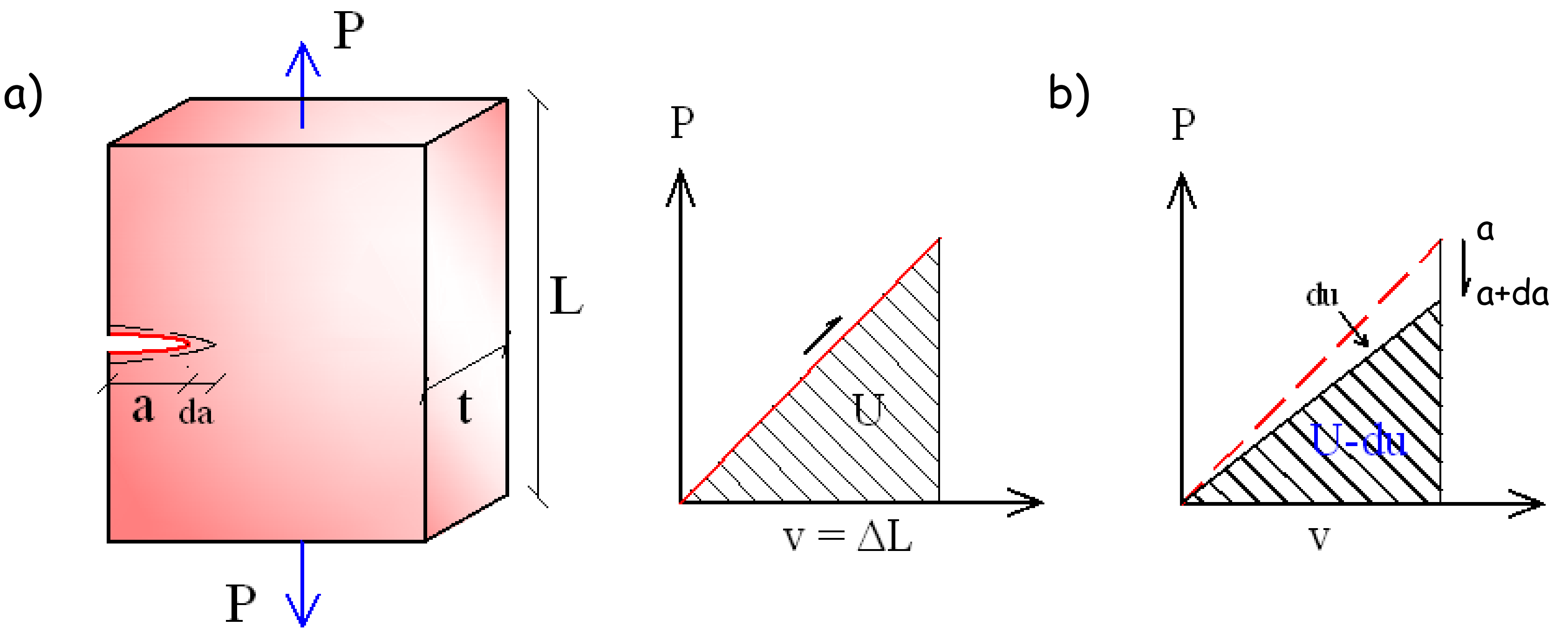


Mode II: Sliding mode
Crack faces slide relative to one another in a direction normal to the leading edge of the crack. Caused by shear loading.



Mode III: Tearing mode
Relative sliding of the crack faces, the direction is parallel to the leading edge. Caused by shear loading.

8.3.1 Strain energy release rate (G) - Griffith Theory



Consider a cracked member under a Mode I force P

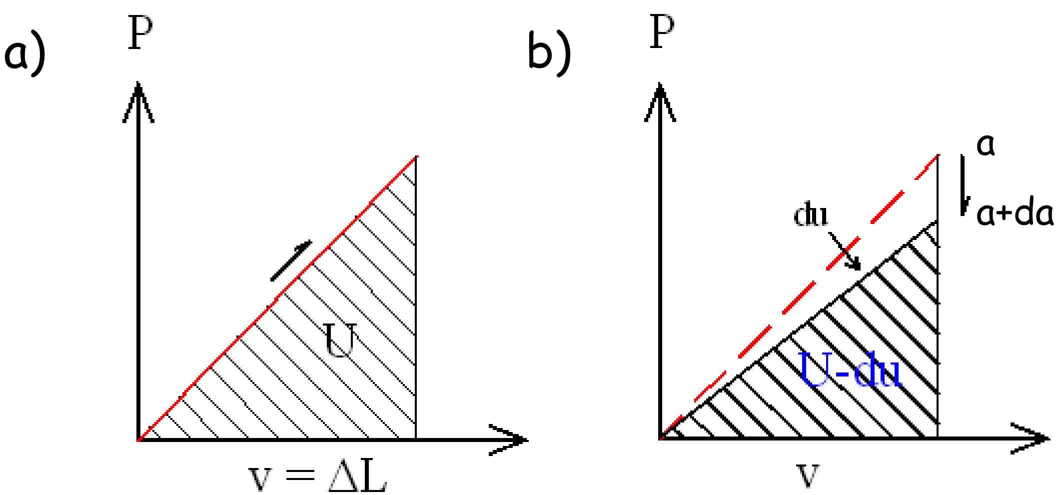
- Crack length is a
- Material behavior is assumed to be linear-elastic (P vs. δ is linear)

8.3.1 Strain energy release rate (G), – Griffith Theory - cont'd

- ✓ Potential energy (U) is stored in the member, as a result of the elastic strains throughout its volume. (v is the displacement at the point of loading and U=Pv/2)

- ✓ If the crack moves ahead by a small amount 'da' while the displacement is held constant, the stiffness of the member decreases as shown by 'b'.

- ✓ This results in the potential energy decreasing by an amount 'dU'; that is, U decreases due to a release of this amount of energy.



8.3.1 Strain energy release rate (G), Griffith Theory, cont'd

The rate of change of potential energy with increase in crack area is defined as the strain energy release rate!

$$G = -\frac{1}{t}\frac{du}{da}$$

Change in the crack area t(da)
Negative sign causes G to have a positive value.

G characterized as the energy per unit crack area required to extend the crack and as such is expected to be the fundamental physical quantity controlling the behavior of the crack.

8.3.1 Strain energy release rate (G) - Griffith theory - cont'd

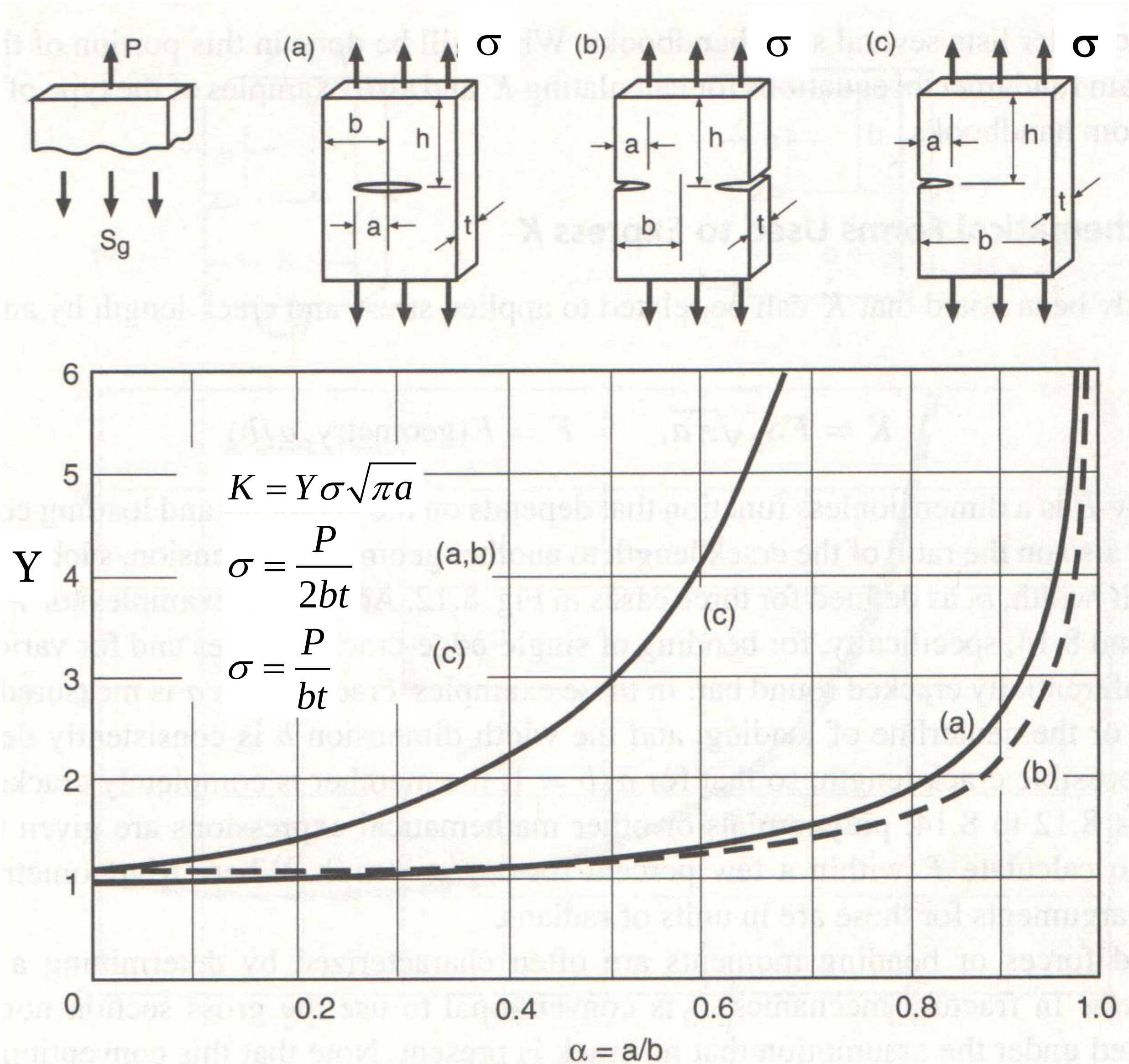
In the original concept proposed by Griffith - All of the potential energy released → was thought to be used in the creation of the new free surface on the crack faces. (True for brittle materials such as glass as tested by Griffith).

Irwin's modification to Griffith theory: In more ductile materials, a majority of the energy may be used in deforming the material in the plastic zone at the crack tip. In applying G to metals in the 1950s, G.R. Irwin showed that the concept was applicable even under these circumstances if the plastic zone was small.

8.3.2 Stress intensity factor, K

K characterizes the magnitude (intensity) of the stresses in the vicinity of an ideally sharp crack tip in a linear-elastic and isotropic material.

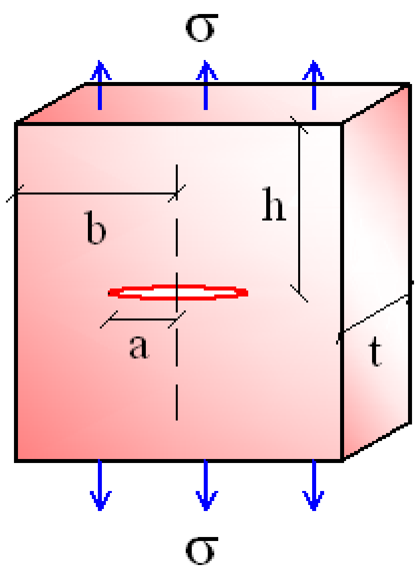
8.4 Application of K to Design and Analysis



K → differs for different crack geometries.

8.4 Application of K to Design and Analysis

K → differs for different crack geometries.



Values for small a/b and limits for 10% accuracy.

a) $K = 1\sigma\sqrt{\pi a}$ $\left(a/b \le 0.4\right)$

b) $K = 1.12\sigma\sqrt{\pi a}$ $\left(a/b \le 0.6\right)$

c) $K = 1.12\sigma\sqrt{\pi a}$ $\left(a/b \le 0.13\right)$

Expression for any $\alpha = a/b$

$$a)\, Y = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}} \quad \left(h/b \ge 1.5\right)$$

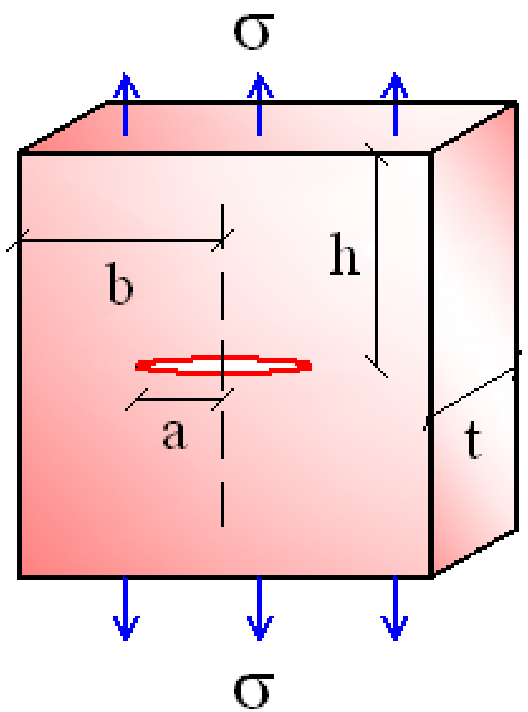
$$b)\, Y = \left(1 + 0.122\cos^4\left(\frac{\pi\alpha}{2}\right)\right)\sqrt{\frac{2}{\pi\alpha}\tan\left(\frac{\pi\alpha}{2}\right)} \quad \left(h/b \ge 2\right)$$

$$c)\, Y = 0.265\left(1-\alpha\right)^4 + \frac{0.857 + 0.265\alpha}{\left(1-\alpha\right)^{3/2}} \quad \left(h/b \ge 1\right)$$

Example 8.1:

A center-cracked plate, as given below, has dimensions b = 50 mm, t = 5 mm and large h; a force of P = 50 kN is applied.

a) What is the stress intensity factor K for a crack length of a = 10mm ?
b) For a = 30mm ?
c) What is the critical length a_c for fracture if the material is 2014-T651 Aluminum ?



Solution:

a) To calculate K for a = 10mm, we need

$$\sigma = \frac{P}{2bt} = \frac{50000N}{2 \times 50mm \times 5mm} = 100MPa$$

$$\alpha = \frac{a}{b} = \frac{10}{50} = 0.2$$

$$\alpha \le 0.4 \, it \, is \, within \, 10\% \, to \, use \, Y = 1$$

$$Thus; \;\; K = \sigma\sqrt{\pi a} = 100MPa\sqrt{\pi\left(0.010m\right)} = 17.7MPa\sqrt{m}$$

b) α =a/b=30 mm/50 mm=0.6
More general expression for Y is needed;

$$Y = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}} = 1.292$$

$$K = Y\sigma\sqrt{\pi a} = 1.292\left(100MPa\right)\sqrt{\pi\left(0.030m\right)} = 39.7MPa\sqrt{m}$$

Cont'd:

$$c)\ K_{IC} = 24MPa\sqrt{m}\ for\ 2014-T651\ Al.$$

a_c is not known Y cannot be determined directly. Assume that $\alpha \lesssim 0.4$ is satisfied, in which case $Y \approx 1$.

$$Then;\quad K_{IC} = \sigma\sqrt{\pi a_c}$$

$$Solving\ for\ a_c\ gives;\quad a_c \approx \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma}\right)^2 = \frac{1}{\pi}\left(\frac{24MPa\sqrt{m}}{100MPa}\right)^2$$

$$a_c = 0.0183m = 18.3mm$$

This corresponds to $\alpha = a_c / b = (18.3mm)/(50mm) = 0.37$ which satisfies $\alpha \lesssim 0.4$, so that the estimated $Y \approx 1$ is acceptable and the result obtained is reasonably accurate.

8.4.3 Safety Factors

$$X_K = \frac{K_{Ic}}{K} = \frac{K_{Ic}}{Y\sigma\sqrt{\pi a}}$$

Another method is to compare the service crack length a with the crack length a_c that is expected to cause failure at the service stress σ.

a_c can be calculated from;

$$K_{IC} = Y_c \sigma \sqrt{\pi a_c}$$

$$X_a = \frac{a_c}{a} = \left(\frac{Y}{Y_c} X_K \right)^2$$

If the crack length expected to occur is relatively small, safety factor against yielding may be calculated simply as follows:

$$X_0 = \frac{\sigma_0}{\sigma}$$

Example 8.2:

Consider the situation of the previous example, where a center-cracked plate of 2014-T651 aluminum, with dimensions b=50mm and t=5mm, is subjected to a force of P=50kN.

a) What is the largest crack length a that can be permitted for a safety factor against fracture of 3.0 in stress?
b) What safety factor on crack length results from the safety factor in stress of (a)?
c) What is the safety factor against yielding?

Fatigue of Materials: Introduction and Stress-Based Approach

Introduction

repeated loads on machines, vehicles etc.

↓

cyclic stress

↓

microscopic physical damage

↓

failure at a stress well below the ultimate stress may occur due to accumulated damage

- This process of damage and failure due to cyclic loading is called ***fatigue***, and it causes an immediate failure.

Approaches to Analyse and Design Against Fatigue Failure.

1. **Traditional Stress – Based Approach**

   Analysis is based on the nominal(average) stresses in the affected region of the engineering component. Nominal stress that can be resisted under cyclic loading is determined by considering mean stresses and by adjusting for the effects of stress raiser, such as cracks, holes etc.

2. **Strain – Based Approach**

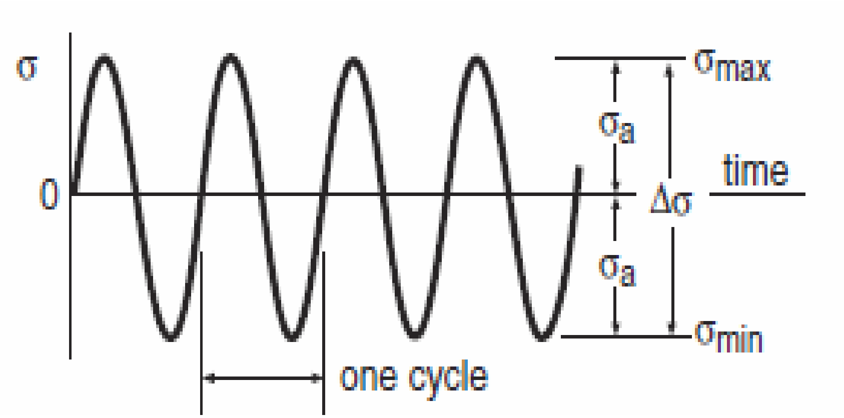
   Involves more detailed analysis of the localized yielding that may occur at stress raisers during cyclic loading.

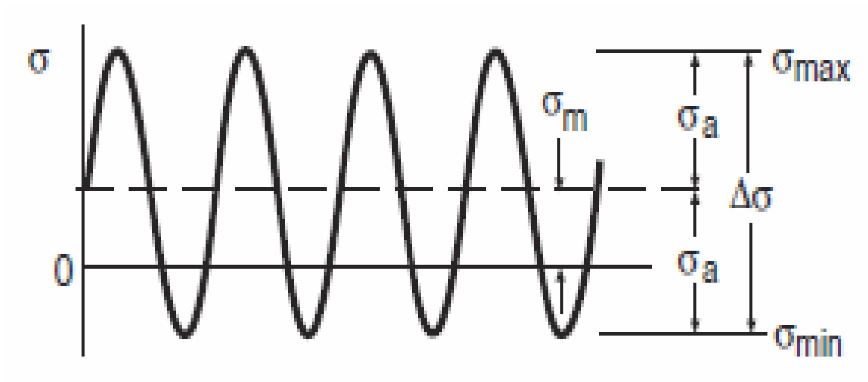
3. **Fracture Mechanics Approach**

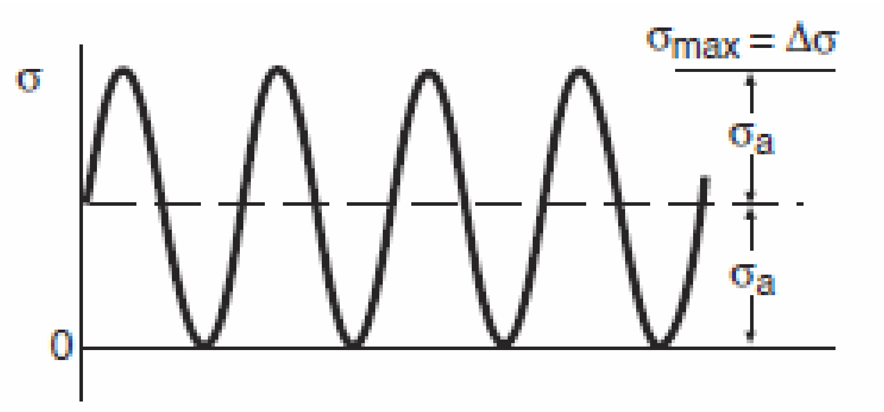
   Specially treats growing cracks by the methods of fracture mechanics.

Stress - Based Approach
Description of Cyclic Loading

<u>Constant Amplitude Stressing:</u> Cycling between maximum and minimum stress levels that are constant.







Stress - Based Approach
Description of Cyclic Loading

- Stress Range: $\Delta\sigma = \sigma_{max} - \sigma_{min}$
- Mean Stress: $\sigma_m = \frac{\sigma_{max}+\sigma_{min}}{2}$
- Stress Amplitude(Alternating stress): $\sigma_a = \frac{\Delta\sigma}{2} = \frac{\sigma_{max}-\sigma_{min}}{2}$
- $\sigma_{max} = \sigma_m + \sigma_a$ $\sigma_{min} = \sigma_m - \sigma_a$

Stress - Based Approach
Description of Cyclic Loading

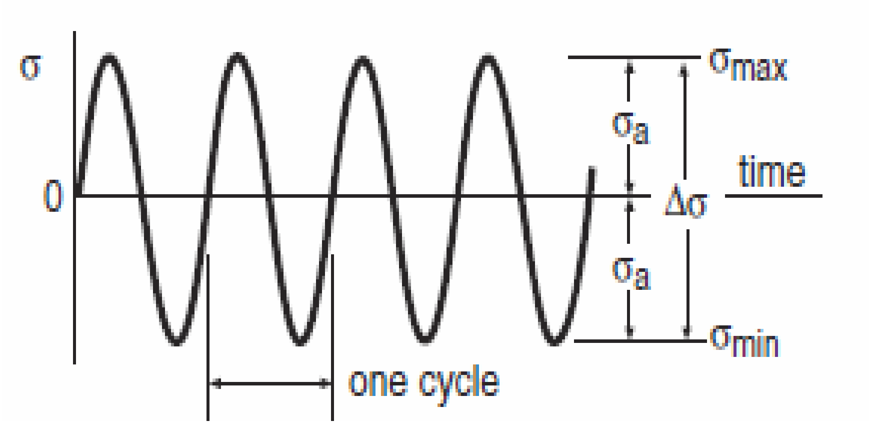
- σ_a *and* $\Delta\sigma$ (always positive, since $\sigma_{max} > \sigma_{min}$)
- Tension (assumed (+)) [$\sigma_{max}, \sigma_{min}$ *and* σ_m can be either positive or negative]
- Stress Ratio: $R = \frac{\sigma_{min}}{\sigma_{max}}$
- Amplitude Ratio: $A = \frac{\sigma_a}{\sigma_m}$
- Derived from the preceeding equation

$$\sigma_a = \frac{\Delta\sigma}{2} = \frac{\sigma_{\max}}{2}(1 - R) \qquad \sigma_m = \frac{\sigma_{\max}}{2}(1 + R) \qquad R = \frac{1 - A}{1 + A} \qquad A = \frac{1 - R}{1 + R}$$

Stress - Based Approach
Description of Cyclic Loading

a) Completely Reversed Stressing

$$\sigma_m = 0, R = -1$$

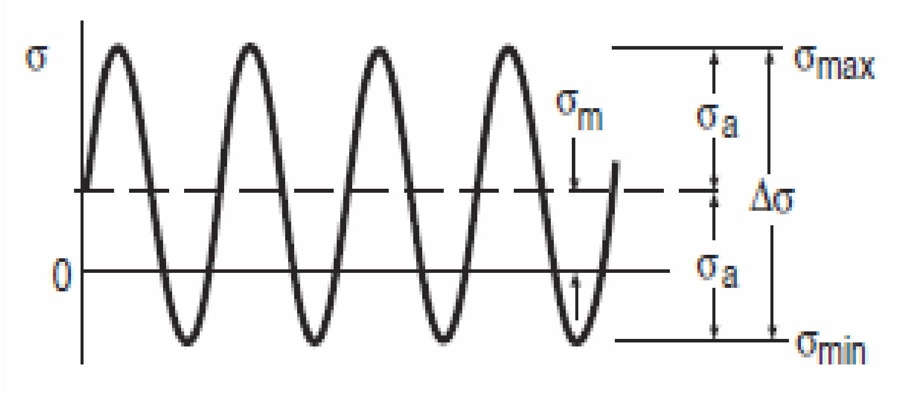


When $\sigma_m = 0$: this loading condition can be specified by giving the amplitude σ_a or the numerically equal σ_{max}

Stress - Based Approach
Description of Cyclic Loading

b) Non Zero Mean Stress

$$\sigma_m \neq 0$$

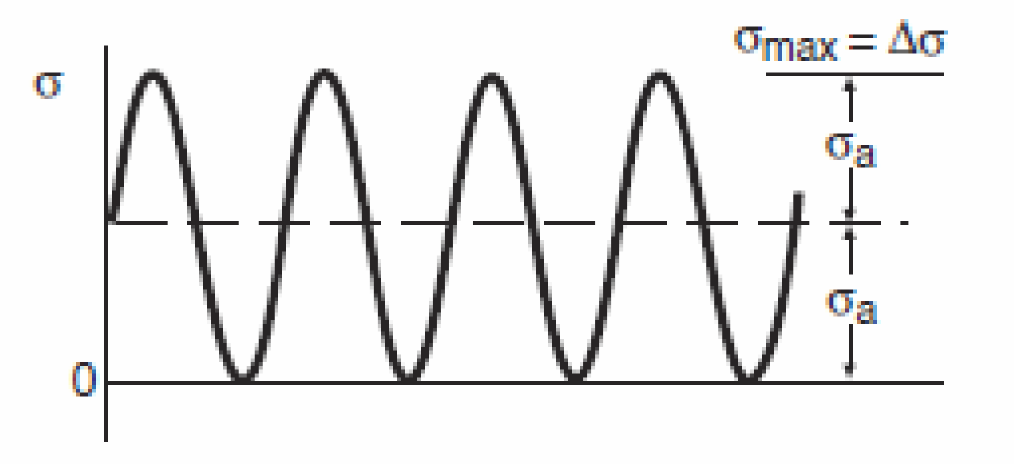


If $\sigma_m \neq 0$: 2 independent values are needed to specify loading. $(\sigma_a \; and \; \sigma_m)$ or $(\sigma_{max} \; and \; R)$ or ($\Delta\sigma$ and R) or $(\sigma_{max} \; and \; \sigma_{min})$ or $(\sigma_a and \; A)$

Stress - Based Approach
Description of Cyclic Loading

c) Zero to Tension Stressing

$\sigma_{min} = 0$ R= 0

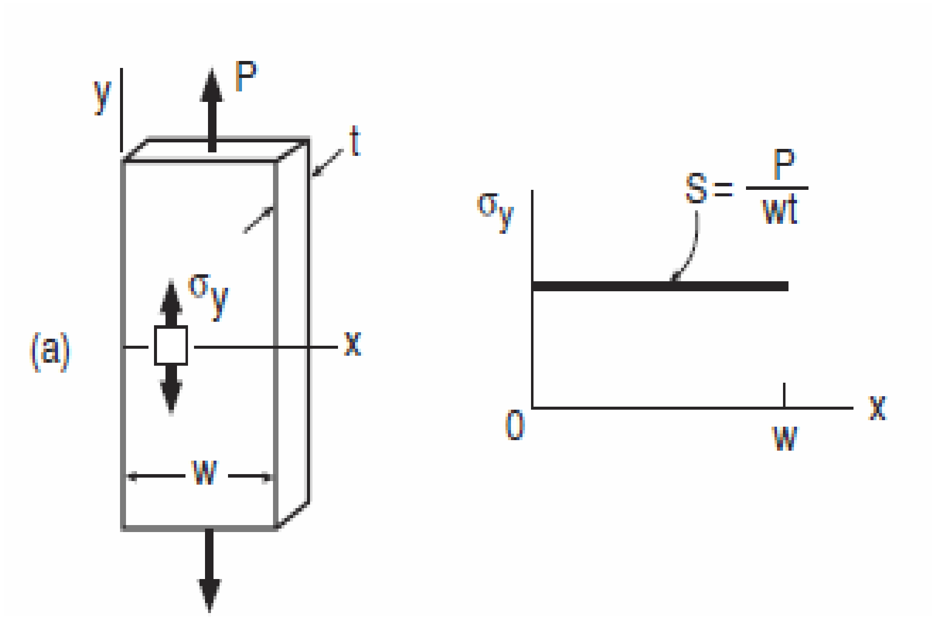


Stress - Based Approach
Point Stresses vs. Nominal Stresses

$\Rightarrow \sigma = stress\ at\ a\ point$ S$\Rightarrow$ $nominal\ or\ average\ stress$

For Axial Loading

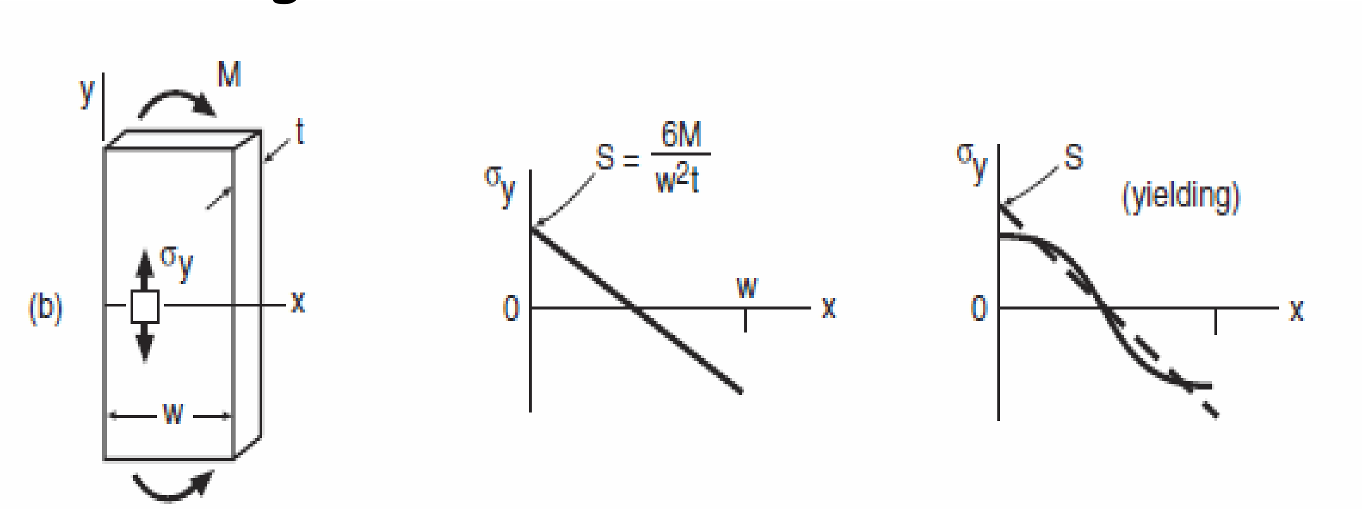
stress is equal everywhere = average stress



Stress - Based Approach
Point Stresses vs Nominal Stresses

$\Rightarrow \sigma = stress\ at\ a\ point$ S$\Rightarrow$ *nominal or average stress*

For Bending



$S = \frac{M * c}{I}$

I: Moment of inertia around the bending axis

c: distance from the neutral axis to the edge

σ=S at the edge

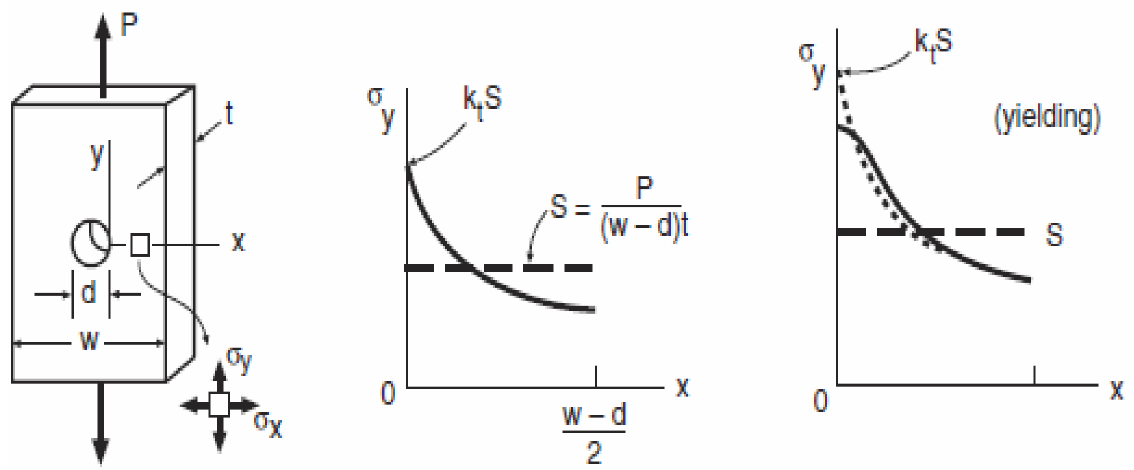
If yielding occurs the actual stress distribution becomes non-linear, σ=S at the edge

Stress - Based Approach
Point Stresses vs Nominal Stresses

$\Rightarrow \sigma = stress\ at\ a\ point$ S$\Rightarrow$ $nominal\ or\ average\ stress$

Notched Member

Nominal stress calculated from the net area after removal of the notch



$\Rightarrow$ S needs to be multiplied by an elastic stress concentration factor(k_t), to obtain the peak stress at the notch $\sigma = k_t$*S

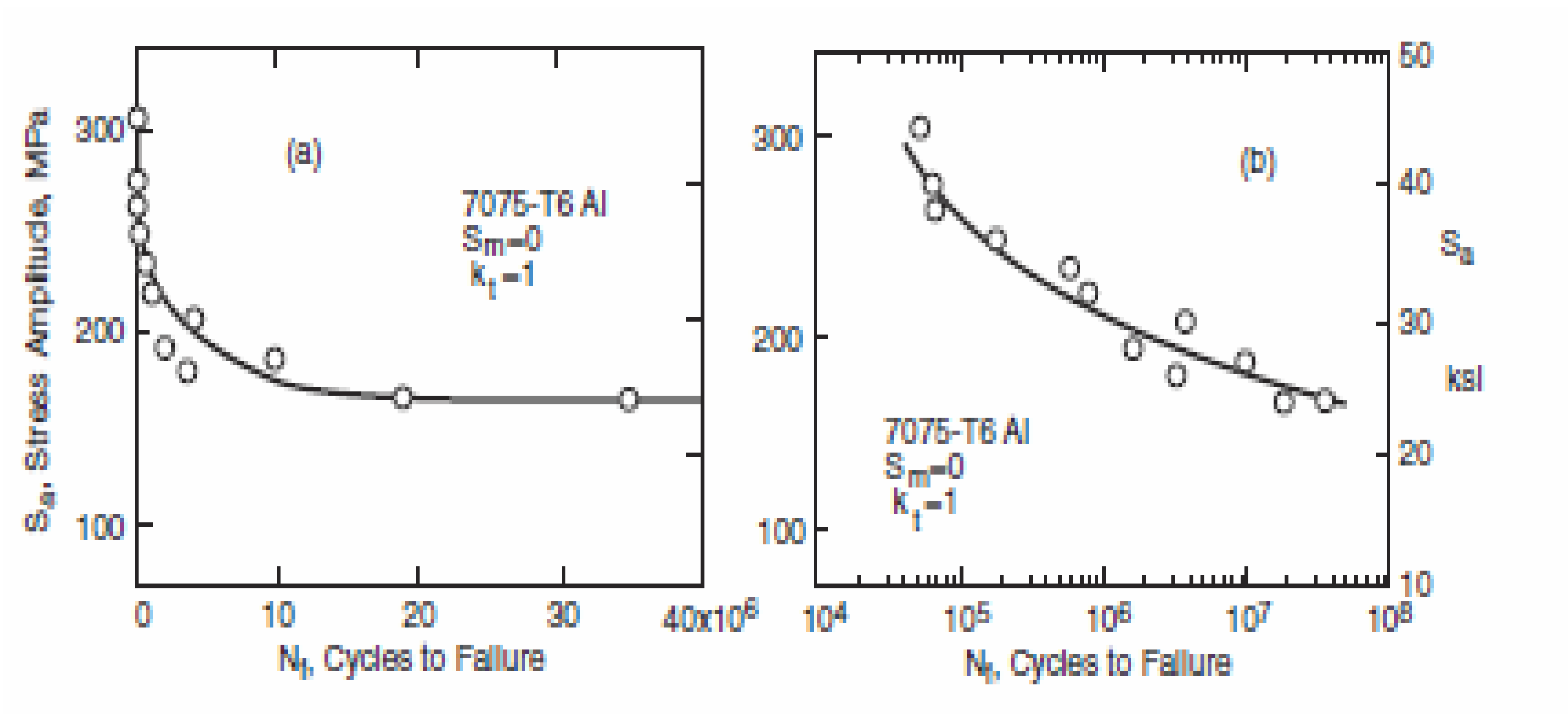
$\Rightarrow$ If yielding occurs k_t does not apply where yielding occur, even locally at the notch, the actual stress σ is lover than k_t*S

Stress vs. Life (S-N) Curves

Severe cyclic loading(**S_1**) with **X** cycle => Specimen => Failure
Severe cyclic loading(**S_2**) with **Y** cycle => Specimen => Failure
S_1 > **S_2** and **X** < **Y**

Stress vs Life curves were obtained from the results of such tests

Stress vs. Life (S-N) Curves



Stress versus life (*S-N*) curves from rotating bending tests of unnotched specimens of an aluminum alloy. Identical linear stress scales are used, but the cycle numbers are plotted on a linear scale in (a), and on a logarithmic one in (b). (Data from [MacGregor 52].)

Stress vs Life (S-N) Curves

If S-N data found to approximate a straight line on a log-linear plot, the following equation can be fitted to obtain a mathematical representation of the curve

$\sigma_a = C + D * \log N_F$ C and D are fitting constants

For data approximating a log-log plot the corresponding equation

$\sigma_a = A * N_f{}^B$ *or* $\sigma_a = \sigma_f{}'(2 * N_f)^b$

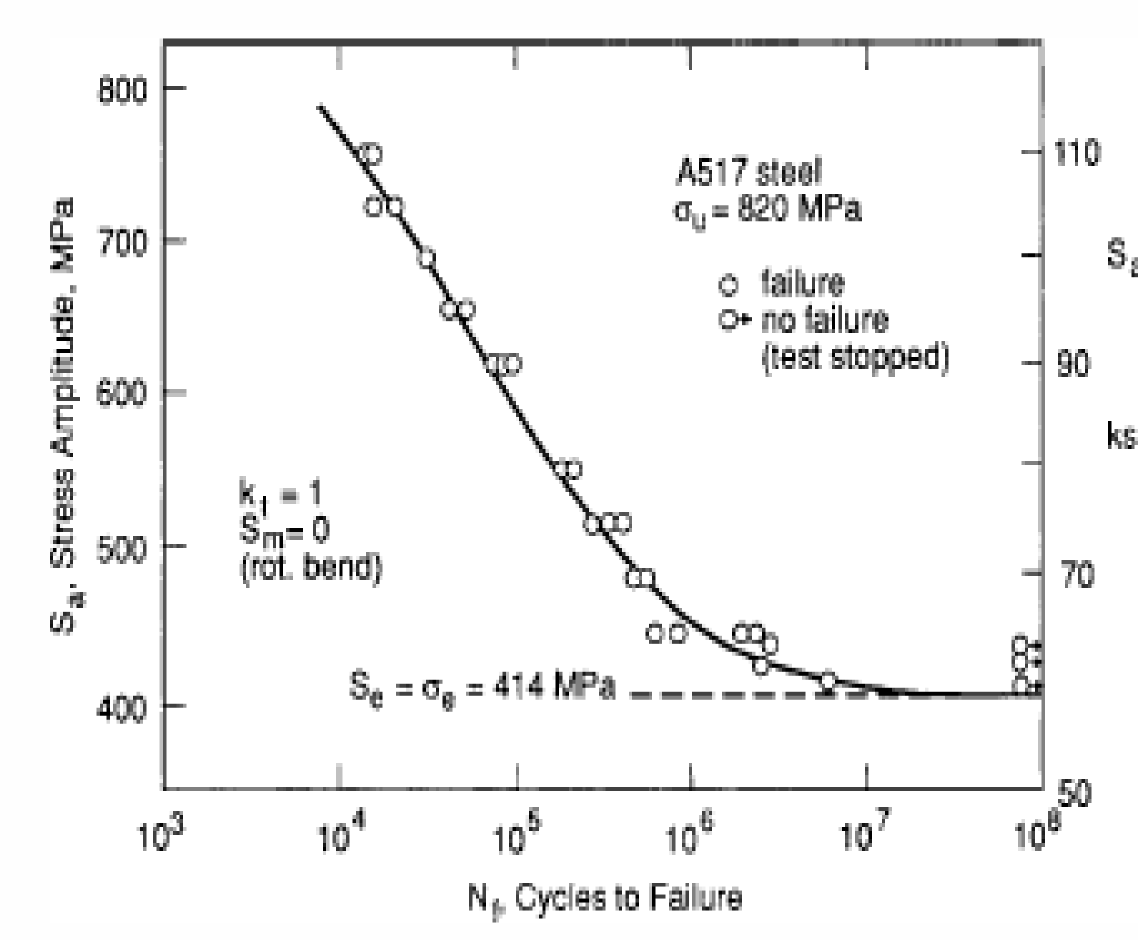
($\sigma_f{}'$ and b related with material properties)

The fitting constants for the 2 forms are related by

$A = 2^b * \sigma_f{}'$ $B = b$

Stress vs Life (S-N) Curves

In some materials, notably plain-carbon and low-alloy steels, there appears to be a distinct stress level below which fatigue failure does not occur under ordinary conditions



S-N curve appears to become flat and asymptotically approach the stress amplitude labeled Se. Such lower limiting stress amplitudes are called fatigue limits or endurance limits.

Stress vs Life (S-N) Curves

The term fatigue strength is used to specify a stress amplitude value from an S-N curve at a particular life of interest. Hence, the fatigue strength at 10^5 cycles is simply the stress amplitude corresponding to $N_f = 10^5$. Other terms used with S-N curves include high-cycle fatigue and low-cycle fatigue.

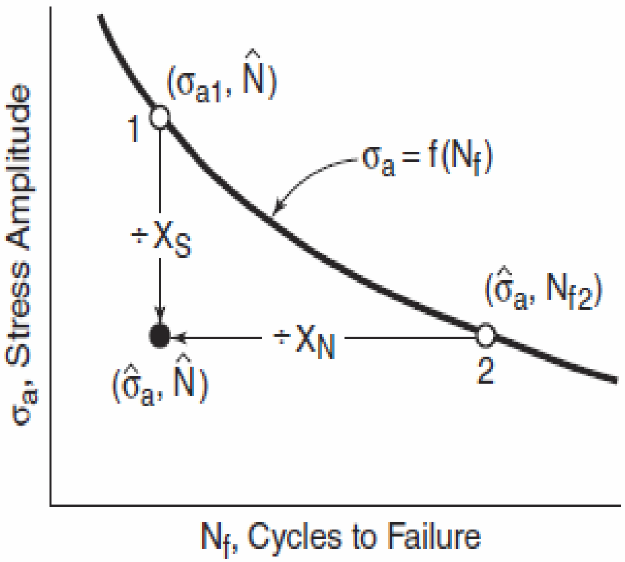
- High cycle fatigue life => the stress is sufficiently low that yielding effects do not dominate the behaviour (Require more than 10^2 to 10^4 cycles)
- Low cycle fatigue life => deals with the effects of plastic deformation

↓

High stress levels, low number of cycles to fatigue.
The material repeatedly plastically deformed

Safety Factors for S – N Curves

$\hat{\sigma}_a$ => stress level and
$\hat{N}$ => number of cycles that are expected to occur in actual service life



This combination must fall below the stress – life curve σ_a=f(N_f) => corresponds to failure

Point 1: σ_{a1} corresponds to failure at desired service life $\hat{N}$

$$X_S = \frac{\sigma_{a1}}{\hat{\sigma}_a} \qquad (N_f = \hat{N})$$

Point 2: Failure life N_{f2} corresponds to the service stress $\hat{\sigma}_a$ comparing N_{f2} with the service life $\hat{N}$ gives the safety factor in life

$$X_N = \frac{N_{f2}}{\hat{N}} \qquad (\sigma_a = \hat{\sigma}_a)$$

Safety Factors for S – N Curves

Safety factors in stress => X_S= 1,5 to 3,0
Safety factors in life => X_N=5 to 20 or more

Since fatigue lives are quite sensitive to the value of stress

$$\sigma_{a1} = A * \hat{N}^{B}$$

$$\hat{\sigma}_a = A * N_{f2}^{B}$$

$$X_S = \frac{A * \hat{N}^{B}}{A * N_{f2}^{B}} = \left(\frac{1}{X_N}\right)^{B} = X_N^{-B} \Rightarrow X_N = X_N^{-1/B}$$

A given safety factor in stress corresponds to a particular safety factor in life

Sources of Cyclic Loading

Loads on components of machines, vehicles, and structures can be divided into four categories, depending on their source:

1. Static Loads: do not vary and continuously present
2. Working Loads: change with time and are incurred as a result of the function performed by component
3. Vibratory Loads: relatively high-frequency cyclic loads that arise from the environment or as a secondary effect of the function of the component(often caused by the fluid turbulence or by the roughness of solid surfaces in contact with one another)
4. Accidental Loads: rare events that do not occur under the normal circumstances

Factors That Affect Fatigue Life

- Mean Stress
- Member Geometry
- Chemical Environment
- Temperature
- Cyclic Frequency
- Residual Stress